

PAPER FOLDING, RIEMANN SURFACES, AND CONVERGENCE OF PSEUDO-ANOSOV SEQUENCES

ANDRÉ DE CARVALHO AND TOBY HALL

ABSTRACT. A method is presented for constructing closed surfaces out of Euclidean polygons with infinitely many segment identifications along the boundary. The metric on the quotient is identified. A sufficient condition is presented which guarantees that the Euclidean structure on the polygons induces a unique conformal structure on the quotient surface, making it into a closed Riemann surface. In this case, a modulus of continuity for uniformizing coordinates is found which depends only on the geometry of the polygons and on the identifications. An application is presented in which a uniform modulus of continuity is obtained for a family of pseudo-Anosov homeomorphisms, making it possible to prove that they converge to a Teichmüller mapping on the Riemann sphere.

1. INTRODUCTION

This article addresses the classical problem of constructing surfaces out of subsets of the plane by making identifications along the boundary: in contrast to the usual discussion arising in the classification of surfaces, however, infinitely many identifications are allowed. The topological structure of the identification space S is studied and conditions are given which guarantee that S is a closed surface. The quotient metric on S induced by the Euclidean metric on the plane is identified, and the question of whether or not this metric induces a unique complex structure on S is discussed. A sufficient condition for uniqueness of the complex structure is given and, when it holds, a modulus of continuity for uniformizing coordinates is obtained. The interplay between the metric and conformal structures is central to the paper, promoting the topological structure to a Riemann surface structure and providing quantitative control over the quotient map. This analytic control is then used to prove convergence of a certain sequence of pseudo-Anosov homeomorphisms to a *generalized* pseudo-Anosov homeomorphism.

Let P be a finite collection of disjoint polygons in the (complex) plane. A *paper-folding scheme* is an equivalence relation which glues together segments — possibly infinitely many — along the boundary of P . The image of ∂P in the quotient space S is called the *scar*: it contains *cone points*, where the total angle is not equal to 2π , and *singular points*, such as accumulations of cone points. The following statements summarize the main theorems of this article.

Topological Structure Theorems (Theorems 42 and 50) *Necessary and sufficient conditions are given for the quotient space S of a paper-folding scheme to be a closed surface. In particular, if the paper-folding scheme is plain (P is a single polygon and the identifications along its boundary are unlinked) then S is a topological sphere and the scar is a dendrite.*

Metric Structure Theorem (Theorem 56) *The quotient metric on S is intrinsic, and S is a conic-flat surface (that is, it is locally isometric to cones on circles) away from singular points.*

Conformal Structure Theorem (Theorem 59) *The natural conformal structure on the conic-flat part of S extends uniquely across an isolated singularity provided that a certain integral diverges. In particular, if there are only finitely many singular points at each of which the relevant integral diverges, then the conformal structure extends uniquely across the singular set, making S into a closed Riemann surface.*

It is possible to extend this theorem to the case of arbitrary singular sets, and this will be the subject of a forthcoming paper.

If the conditions of both Theorem 50 and Theorem 59 are satisfied, then S has a natural closed Riemann surface structure. If S is a topological sphere, as is the case for plain paper foldings, then S is isomorphic to the Riemann sphere. A modulus of continuity for a suitably normalized uniformizing map from the polygon to the Riemann sphere is found and the following theorem is proved:

Modulus of continuity of uniformizing map (Theorem 84) *The uniformizing map has a modulus of continuity which depends only on the geometry of the polygon and on the metric on the scar.*

The theorems above belong to the fields of surface topology, Riemann surface theory, and geometric function theory. The question that motivated this work, however, comes from dynamical systems theory and was described to the first author many years ago by Dennis Sullivan. Consider a self-homeomorphism of a surface, such as the much-studied Hénon diffeomorphism of the plane

$$f(x, y) := (a - x^2 - by, x),$$

where a, b are real parameters. There is a large set of parameters for which f is *chaotic* which means, amongst other things, that it has infinitely many periodic orbits. If Q is a finite f -invariant set (a union of periodic orbits), consider the isotopy class of f in the punctured plane $\mathbb{R}^2 \setminus Q$. Thurston's classification of surface homeomorphisms provides a canonical representative φ_Q in this isotopy class, which is, typically, a *pseudo-Anosov* homeomorphism. If Q_n is an increasing sequence which exhausts the collection of finite f -invariant sets, one can imagine passing to the limit in the sequence φ_{Q_n} of associated pseudo-Anosov maps. The resulting limit would be a *tight* representative of f which contains 'all' of its dynamics. The paper ends with an application of the theorems above to establish a case of this scenario:

Convergence to the tight horseshoe (Theorem 105) *There exists a cofinal sequence in the set of horseshoe braid types partially ordered by forcing whose associated pseudo-Anosov homeomorphisms converge to the tight horseshoe.*

To understand the existence of limits of sequences of pseudo-Anosov maps on a fixed surface S relative to varying finite subsets Q has several consequences. From a purely intrinsic point of

view, it is interesting to be able to describe the closure of the set of relative pseudo-Anosov maps on S — an important class of homeomorphisms — in the space of all self-homeomorphisms of S . Also, Thurston’s classification of surface homeomorphisms up to isotopy produces geometrically and conformally rigid structures out of the topology. Understanding limits of pseudo-Anosovs would do the same in a more general context and, in this case, the dynamics and not the topology would be the driving force. It is hoped that this will eventually lead to the correct 2-dimensional analogue of the Milnor-Thurston theorem stating that every multimodal interval map has a piecewise-linear quotient with same entropy. There are also connections to be explored with infinite dimensional Teichmüller theory, since the tight limits are Teichmüller mappings with respect to quadratic differentials in the L^1 -closure of the set of meromorphic quadratic differentials.

Section 2 provides a brief summary of results from surface topology, metric geometry, and geometric function theory which will be used in the remainder of the paper. In Section 3 the main objects of study, paper-folding schemes, are defined. These are identification schemes around the boundaries of finite disjoint unions of polygons, whose (metric) quotients are called *paper spaces*.

Sections 4 and 5 contain discussions of the topological, metric, and conformal structures of paper spaces. To simplify the exposition, some of the arguments are first developed in the simplest and most common case of plain paper foldings, before being extended to the general case.

In Section 6, moduli of continuity are discussed, first locally for uniformizing coordinates charts on the neighborhood of points on S at which the conformal structure extends uniquely and then globally in the case where S is a complex sphere and is thus isomorphic to the Riemann sphere.

In Section 7, the dynamical application is given. A sequence of pseudo-Anosov maps of the punctured sphere is considered: each can be regarded as a quasi-conformal homeomorphism of an appropriate (finite) paper surface. Using the results of Section 6, the natural limit of this sequence is constructed. In fact this is equivalent to taking the limit of pseudo-Anosov maps over an exhaustion of the collection of finite invariant sets of Smale’s horseshoe map.

A brief account of the results described in this article, their application, and an outline of the methods of proof can be found in [7].

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2. BACKGROUND

This section contains a summary of background theory which will be used in the article, together with references to more detailed accounts. Section 2.1 contains topological results which will be used in Section 4 to give conditions under which the quotient space associated with a paper-folding scheme is a topological sphere or, more generally, a surface. The main tool here is Moore’s theorem [14, 15], which provides conditions under which the quotient of a sphere by an equivalence relation is again a sphere. Section 2.2 contains a brief summary of the theory of quotient metric spaces and their relationship to topological quotients. Finally, Section 2.3 describes some results

from geometric function theory which will be used in Section 5 to give conditions under which the conformal structure on the quotient space extends uniquely across an isolated singularity; and in Section 6 to determine a modulus of continuity for a uniformizing map from the polygon to the Riemann sphere in the case of a plain paper folding.

2.1. Moore's theorem and the dendrite quotient theorem.

Definitions 1 (Separation, continuum). A *separation* of a topological space X is a decomposition of X as a disjoint union $X = A \cup B$ of non-empty closed subsets of X . The space X is *connected* if no separation of X exists.

A set $C \subset X$ *separates* two points $x, y \in X$ (respectively subsets $D, E \subset X$) if there is a separation $X \setminus C = A \cup B$ with $x \in A, y \in B$ (respectively $D \subset A, E \subset B$). If X is connected and $X \setminus C$ is not, C *separates* X .

A *continuum* is a compact connected (subset of a) Hausdorff topological space.

Definitions 2 (Monotone upper semi-continuous decomposition). An equivalence relation \sim on a topological space X is *closed* if it is closed as a subset of $X \times X$. (On compact Hausdorff spaces, being closed is the same as saying that $x_n \sim y_n, x_n \rightarrow x$ and $y_n \rightarrow y$ imply $x \sim y$; and an equivalence relation is closed if and only if the quotient space is Hausdorff.)

A *decomposition* of X is synonymous with a partition of X , i.e., a collection of disjoint subsets whose union is X . A decomposition \mathcal{G} of a topological space into compact subsets is *upper semi-continuous* (*usc*) if the associated equivalence relation is closed. A decomposition is *monotone* if its elements are connected.

The expression “monotone upper semi-continuous” will be abbreviated *musc*.

Definition 3 (Realizing an equivalence relation). Let P be a subset of a topological space X , and \sim be an equivalence relation on P . A decomposition \mathcal{G} of X *realizes* \sim if all elements of \mathcal{G} intersect P , and each intersection of an element of \mathcal{G} with P is a \sim -equivalence class. (This means that P/\sim is naturally identified with X/\mathcal{G} .)

Definition 4 (Saturation). Let R be a reflexive and symmetric relation on a set X . A subset U of X is *R-saturated* if it contains $\{y \in X : yRx\}$ for all $x \in U$.

Any collection \mathcal{P} of subsets of X (not necessarily a partition) naturally defines a symmetric and reflexive relation R (two distinct points are related if they belong to the same element of \mathcal{P}), which is transitive if the elements of \mathcal{P} are disjoint. A subset U of X is *\mathcal{P} -saturated* if it is R -saturated.

The following theorem, due to Moore [14], is the main tool used for deciding when the quotient of an equivalence relation on the 2-sphere is again the sphere. A generalization for surfaces was given by Roberts and Steenrod [16].

Theorem 5 (Moore on the 2-sphere). *A topological quotient of the 2-sphere by a musc decomposition whose elements do not separate it is again homeomorphic to the 2-sphere.*

The topological structure of the scar of a paper-folding scheme will also be important, and the main tool which will be used in this regard is Theorem 7 below.

Definitions 6 (Dendrite, dendritic collection). A *dendrite* is a locally connected continuum which does not contain any simple closed curve. A *local dendrite* is a continuum which is locally a dendrite, i.e., for which every point has a (closed) neighborhood which is a dendrite.

A collection \mathcal{G} of disjoint subsets of a topological space X is *non-separated* if no element of \mathcal{G} separates two points of any other single element of \mathcal{G} . A non-separated collection \mathcal{G} is *dendritic* if given $g \in \mathcal{G}$ and $y \notin g$, there exists $g' \in \mathcal{G}$ which separates y and g .

The following theorem can be found in [19]. There the word *saturated* is used to signify what was defined as *dendritic* above (the former word has already been used here).

Theorem 7 (Dendrite quotient). *Let \mathcal{G} be a dendritic decomposition of a continuum X . Then \mathcal{G} is usc and the quotient space X/\mathcal{G} is a dendrite.*

Here is a list of properties of dendrites which will be used later. The main references for them are [11, 19].

Theorem 8 (Properties of dendrites).

- a) *Any two distinct points in a dendrite G are separated by a third point in G . Conversely, any continuum with this property is a dendrite.*
- b) *If G is a dendrite and $x, y \in G$, there exists a unique arc $\gamma \subset G$ whose endpoints are x, y . The notation $[x, y]_G$ will be used to indicate this arc and expressions such as $(x, y]_G$ will have the usual meaning (i.e., $(x, y]_G = [x, y]_G \setminus \{x\}$).*
- c) *A point p in a continuum G is an endpoint if it has arbitrarily small neighborhoods (in G) whose boundary is a single point; and it is a cut point if $G \setminus \{p\}$ is disconnected. A continuum is a dendrite if and only if all of its points are either endpoints or cut points.*
- d) *Dendrites are contractible [9].*
- e) *Every subcontinuum of a dendrite is also a dendrite. (A subcontinuum of a continuum G is a subset which is also a continuum.)*
- f) *Every connected subset of a dendrite is arcwise connected, and the intersection of any two connected subsets is connected.*

2.2. Quotient metric spaces and intrinsic metrics. The quotient spaces associated with paper-folding schemes are metric rather than topological quotients, and this section summarizes some relevant definitions and results. [3] is an excellent reference for readers seeking further details.

Definitions 9 (Metric and semi-metric). A *metric* on a set X is a function $d_X : X \times X \rightarrow \mathbb{R} \cup \{+\infty\}$ satisfying the usual conditions of positivity, symmetry and the triangle inequality (it is convenient to allow two points to be infinitely distant from one another). When no ambiguity arises, subscripts may be dropped so that, for example, the usual Euclidean distance on \mathbb{R}^2 or \mathbb{C} may be denoted $d_{\mathbb{R}^2}(x, y)$, $d_{\mathbb{C}}(x, y)$, $d(x, y)$, or even $|xy|$.

A *semi-metric* on a set X satisfies the same axioms as a metric, except that distinct points are allowed to be at semi-distance zero from one another.

Notation 10. The following notation is used for metric and semi-metric spaces (X, d_X) . If $x \in X$ and $r \geq 0$, then

$$(1) \quad \begin{aligned} B_X(x; r) &:= \{y \in X : d_X(y, x) < r\}, \\ \overline{B}_X(x; r) &:= \{y \in X : d_X(y, x) \leq r\}, \quad \text{and} \\ C_X(x; r) &:= \{y \in X : d_X(y, x) = r\}. \end{aligned}$$

If ambiguity lurks, subscripts may be enhanced so that $B_X(x; r)$, for example, may be denoted $B_{(X, d_X)}(x; r)$ or $B_{d_X}(x; r)$. At the other extreme, if the ambient space is clear, $B(x; r)$ may be used.

If $x \in X$ and $A \subset X$, the distance from x to A is

$$d_X(x, A) := \inf\{d_X(x, y) : y \in A\}.$$

and if $A, B \subset X$, the distance between them is

$$d_X(A, B) := \inf\{d_X(x, y) : x \in A, y \in B\}.$$

Given a set $A \subset X$, the sets $B_X(A; r)$, $\overline{B}_X(A; r)$ and $C_X(A; r)$ are defined by substituting $d(y, A)$ for $d(y, x)$ in (1).

Definitions 11 (Quotient metric). Let (X, d) be a metric space and R be a reflexive and symmetric relation on X (which in this article will usually arise from a collection \mathcal{P} of subsets of X , as in Definition 4). An *R-chain* from x to y is a sequence $((p_i, q_i))_{i=0}^k$ in X^2 such that xRp_0 , q_iRp_{i+1} for $i = 0, \dots, k-1$, and q_kRy . Its *length* is

$$L^R((p_i, q_i)) := \sum_{i=0}^k d(p_i, q_i)$$

(thus one pays to move between unrelated elements of X , but moving between related elements is free).

Define $d^R : X \times X \rightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$ by

$$d^R(x, y) := \inf \{L^R((p_i, q_i)) : ((p_i, q_i)) \text{ is an } R\text{-chain from } x \text{ to } y\}.$$

Then d^R is a semi-metric on X . The equivalence relation which identifies points at d^R -semi-distance 0 is denoted \sim_R and the quotient space under this equivalence relation is the *quotient metric space* of (X, d) under the relation R , denoted either $(X/d^R, d^R)$ or $(X/\sim_R, d^R)$. (So the same symbol d^R is used to denote both the semi-metric and the metric on the quotient space.)

Many mathematicians, including the authors, are more familiar with topological quotients. Assuming that R is an equivalence relation, there are essentially two reasons why the quotient metric space $(X/d^R, d^R)$ and the topological quotient X/R may differ. First, they may differ as sets: for example, the metric quotient of \mathbb{R} by \mathbb{Q} is a point (as is the metric quotient of any metric space by a relation R which identifies a dense set of points), since there are arbitrarily short R -chains

joining any two points. Second, even if the two quotients agree as sets, they may have different topologies. An instructive example (Example 3.1.17 of [3]) is the following. Let X be a disjoint union of countably many intervals I_i , of lengths ℓ_i , and consider the quotients by the relation which identifies the left hand endpoints of all of the intervals. The topology of the metric quotient depends on the lengths ℓ_i : in particular, it is compact if $\ell_i \rightarrow 0$, and non-compact otherwise. Clearly, though, the topological quotient does not depend on these lengths.

The case where $\ell_i \rightarrow 0$ will be of interest later:

Definition 12 (∞ -od). An ∞ -od is the metric quotient of countably many intervals I_i of lengths ℓ_i , with $\ell_i \rightarrow 0$ as $i \rightarrow \infty$, under the relation which identifies all of their left hand endpoints.

The problem of having quotients which agree as sets but not as topological spaces does not arise if X is compact (see [3]):

Theorem 13 (Metric and topological quotients). *Let X be a compact metric space, and R be a reflexive and symmetric relation on X . Then the topological quotient X/\sim_R and the metric quotient $(X/\sim_R, d^R)$ are homeomorphic.*

(Notice that in this statement the two quotients are equal as sets by definition, since the same equivalence relation \sim_R is used in both cases.)

The quotient associated to a paper-folding scheme on a polygon P is constructed by taking the metric quotient of P by a relation R determined by the folding scheme. Since P is compact, this metric quotient is homeomorphic to the topological quotient P/\sim_R , which will be studied using Moore's theorem.

In this article subsets of metric spaces will normally be endowed not with the subspace metric, but with the intrinsic metric inherited from the parent space. The remainder of this section provides a brief summary of these ideas.

Definitions 14 (Length and intrinsic metric). Let (X, d) be a metric space. A *path* in X is a continuous map $\gamma: [a, b] \rightarrow X$. The *length* of the path γ is

$$|\gamma|_X = \sup \left\{ \sum d(\gamma(t_i), \gamma(t_{i+1})) \right\} \in \mathbb{R}_{\geq 0} \cup \{\infty\},$$

where the supremum is taken over all finite partitions $a = t_0 < t_1 < \dots < t_k = b$ of the interval $[a, b]$. When the metric space X is clear from the context, the length will be denoted $|\gamma|$ without the subscript. If γ is injective, then the length of the image of γ is defined to be $|\text{im}(\gamma)|_X := |\gamma|_X$.

A path is *rectifiable* if its length is finite.

A metric is *intrinsic* if the distance between any two points is arbitrarily well approximated by lengths of curves joining the two points, that is,

$$d(x, y) = \inf \{ |\gamma| : \gamma: [a, b] \rightarrow X \text{ is a path with } \gamma(a) = x, \gamma(b) = y \}.$$

The metric is *strictly intrinsic* if the infimum is attained, that is, if for every $x, y \in X$, there exists a continuous path from x to y whose length equals $d(x, y)$. (In particular, $d(x, y) = +\infty$ if x and y

lie in different path components of X .) It can be shown (see [3]) that a complete locally compact intrinsic metric is strictly intrinsic. In particular,

Theorem 15. *A compact intrinsic metric is strictly intrinsic.*

If d is not an intrinsic metric, then there is an *induced intrinsic metric* \widehat{d} on X , defined by

$$\widehat{d}(x, y) = \inf \{|\gamma| : \gamma : [a, b] \rightarrow X \text{ is a path with } \gamma(a) = x, \gamma(b) = y\}$$

(that \widehat{d} is an intrinsic metric follows from the straightforward observation that the length $|\gamma|$ doesn't depend on whether the metric d or \widehat{d} is used). For example, if $P \subset \mathbb{R}^2$ is a polygon, then there is an intrinsic metric d_P on P induced by the metric on \mathbb{R}^2 , which does not agree with the subspace metric on P unless P is convex.

A proof of the following result can be found on pp. 62–63 of [3].

Lemma 16. *Let (X, d) be an intrinsic metric space, and R be a reflexive and symmetric relation on X . Then the quotient metric space $(X/d^R, d^R)$ is also intrinsic.*

Remark 17. The projection π from a metric space X to a metric quotient X/d^R does not increase the distances between points or the lengths of paths. That is, if $x, y \in X$ then $d^R(\pi(x), \pi(y)) \leq d(x, y)$; and if γ is a path in X , then $|\pi \circ \gamma|_{X/d^R} \leq |\gamma|_X$.

The following three results about Hausdorff 1-dimensional measure μ_X^1 on a metric space X are Lemma 2.6.1 of [3], Theorem 2.6.2 of [3], and a corollary of Theorem 29 of [17] respectively.

Lemma 18. *If X is a connected metric space, then $\mu_X^1(X) \geq \text{diam } X$.*

Lemma 19. *If $\gamma : [a, b] \rightarrow X$ is a rectifiable simple path, then $|\gamma|_X = \mu_X^1(\gamma([a, b]))$.*

Lemma 20. *Let (X, ρ) and (Y, σ) be metric spaces, and $f : X \rightarrow Y$ be a function with $\sigma(f(x), f(y)) \leq \rho(x, y)$ for all $x, y \in X$. Then $\mu_Y^1(f(A)) \leq \mu_X^1(A)$ for every μ_X^1 -measurable subset A of X .*

2.3. Background geometric function theory. In this section some results from geometric function theory which will be used in Sections 5 and 6 are summarized. Standard references are Ahlfors-Sario [2], Ahlfors [1] and Lehto and Virtanen [12].

Definitions 21 (Module of an annular region, concentric, nested). An *annular region* in a surface is a subset homeomorphic to an open round annulus

$$A(r_1, r_2) = \{z \in \mathbb{C} : r_1 < |z| < r_2\},$$

where $0 \leq r_1 < r_2 \leq \infty$. If $R \subset \mathbb{C}$ is an annular region, then there is a conformal map taking R onto some round annulus $A(r_1, r_2)$, and this map is unique up to postcomposition with a homothety. It follows that the ratio r_2/r_1 is a conformal invariant of R , and the *module* $\text{mod } R$ of R is defined by

$$\text{mod } R = \begin{cases} \infty & \text{if } r_1 = 0 \text{ or } r_2 = \infty, \\ \frac{1}{2\pi} \ln \frac{r_2}{r_1} & \text{otherwise.} \end{cases}$$

Suppose that S is a closed topological disk and that $R \subset S$ is an annular region. The *bounded component* of $S \setminus R$ is the one which is disjoint from ∂S . Annular regions $R_0, R_1 \subset S$ are *concentric* if the bounded complementary component of one of them is contained in the bounded complementary component of the other. They are *nested* if they are concentric and disjoint, that is, if one is entirely contained in the bounded complementary component of the other.

If R_0, R_1 are concentric annular regions and $R_1 \subset R_0$, then $\text{mod } R_1 \leq \text{mod } R_0$. The way in which this observation will be used here is in the form of the following:

Lemma 22 (Conformal puncture). *Consider a 1-parameter family $R(t)$ ($t \geq 0$) of concentric annular regions contained in a disk in the complex plane. Assume each $R(t)$ has finite module and that $R(t') \subset R(t)$ if $t < t'$. If $\text{mod } R(t) \rightarrow \infty$ as $t \downarrow 0$ then $R(0) = \bigcup_{t>0} R(t)$ is an open annular region whose bounded complementary component is a point.*

To verify the hypotheses of this lemma, the following fundamental inequality will be used.

Lemma 23. *If $\{R_n\}$ is a finite or countable family of nested annular regions all contained in and concentric with the annular region R , then*

$$(2) \quad \text{mod } R \geq \sum \text{mod } R_n.$$

From this it follows that if $\sum \text{mod } R_n$ diverges then at least one of the complementary components of R is a point.

A *conformal metric* on \mathbb{C} is a metric obtained defining the length of arcs by $|\gamma|_\nu := \int_\gamma \nu(z) |dz|$, where ν is a nonnegative Borel measurable function defined on \mathbb{C} . If R is an annular region in \mathbb{C} whose boundary components are C_1, C_2 then

$$\text{mod } R = \sup \left\{ \frac{d_\nu(C_1, C_2)^2}{\text{Area}_\nu(R)} : \nu(z) |dz| \text{ is a conformal metric} \right\},$$

where $d_\nu(C_1, C_2)$ is the ν -distance between the boundary components of R (i.e., the minimum ν -length of an arc with endpoints in C_1 and C_2) and $\text{Area}_\nu(R) = \iint_R \nu(z)^2 dx dy$ is the ν -area of R . It follows that

$$(3) \quad \text{mod } R \geq \frac{d_\nu(C_1, C_2)^2}{\text{Area}_\nu(R)}$$

for any conformal metric $\nu(z) |dz|$.

Definition 24 (Grötzsch annular region). Let $t \in [0, 1]$. The *Grötzsch annular region* $\text{Gr}(t)$ is the region obtained from the open unit disk by removing the closed interval $[0, t]$ in the real axis.

The following inequality can be found in [12] and [1]:

$$(4) \quad \text{mod } \text{Gr}(t) \leq \frac{1}{2\pi} \ln \frac{4}{t}.$$

The following theorem (see [1], p.72) will be of fundamental importance in Section 6.

Theorem 25 (Grötzsch Annulus Theorem). *Let R be an annular region contained in the unit disk in the complex plane. If both 0 and a point z are contained in the bounded complementary component of R , then*

$$\text{mod } R \leq \text{mod Gr}(|z|) \leq \frac{1}{2\pi} \ln \frac{4}{|z|}.$$

The following distortion theorem will be needed. For completeness, a proof is included. The term *univalent* is used in this article as a synonym for *injective holomorphic*.

Theorem 26. *Let $P \subset \mathbb{C}$ be a closed topological disk, $\tilde{p} \in \text{Int}(P)$ and $\Phi: \text{Int}(P) \rightarrow \mathbb{C}$ be a univalent function with $\Phi(\tilde{p}) = 0$ and $\Phi'(\tilde{p}) = 1$. Let $\tilde{Q}(h)$ denote the closed interior collar neighborhood of ∂P of $d_{\mathbb{C}}$ -width $h > 0$ and set $P_h := P \setminus \tilde{Q}(h)$. Assume that h is small enough that P_h is path connected and $\tilde{p} \in P_h$, and set*

$$\kappa := \exp \left(\frac{8 \text{diam}_{P_h} P_h}{h} \right),$$

where $\text{diam}_{P_h} P_h$ denotes the diameter of P_h in the intrinsic metric d_{P_h} on P_h induced by $d_{\mathbb{C}}$. Then for any $\tilde{q} \in P_h$,

$$\frac{1}{\kappa} \leq |\Phi'(\tilde{q})| \leq \kappa.$$

In particular, Φ is κ -biLipschitz in P_h .

Proof. The usual distortion theorem (see [1]) states that if $f: \mathbb{D} \rightarrow \mathbb{C}$ is univalent then, for any $z_1, z_2 \in \mathbb{D}$,

$$(5) \quad \frac{1}{m(z_2, z_1)} \leq \frac{|f'(z_1)|}{|f'(z_2)|} \leq m(z_1, z_2),$$

where

$$(6) \quad m(z_1, z_2) := \frac{1 + |z_1|}{1 - |z_2|} \cdot \left(\frac{1 + |z_2|}{1 - |z_1|} \right)^3.$$

Let $\zeta: \mathbb{D} \rightarrow \text{Int}(P)$ be a Riemann mapping with $\zeta(0) = \tilde{p}$ and set $f := \Phi \circ \zeta$. Then both f and ζ are univalent so that, writing $\zeta_i := \zeta(z_i)$ for $i = 1, 2$, it follows from (5) that

$$\frac{1}{m(z_1, z_2) \cdot m(z_2, z_1)} \leq \frac{|\Phi'(\zeta_1)|}{|\Phi'(\zeta_2)|} = \frac{|f'(z_1)|}{|f'(z_2)|} \cdot \frac{|\zeta'(z_2)|}{|\zeta'(z_1)|} \leq m(z_1, z_2) \cdot m(z_2, z_1),$$

and (6) gives

$$\begin{aligned} m(z_1, z_2) \cdot m(z_2, z_1) &= \left(\frac{1 + |z_1|}{1 - |z_1|} \cdot \frac{1 + |z_2|}{1 - |z_2|} \right)^4 \\ &= [\exp(d_{\mathbb{D}}^H(0, z_1)) \cdot \exp(d_{\mathbb{D}}^H(0, z_2))]^4, \end{aligned}$$

where $d_{\mathbb{D}}^H$ denotes the Poincaré distance in the unit disk \mathbb{D} . Setting $z_2 = 0$ it follows that, for any $z \in \mathbb{D}$,

$$(7) \quad [\exp(d_{\mathbb{D}}^H(0, z))]^{-4} \leq |\Phi'(\zeta(z))| \leq [\exp(d_{\mathbb{D}}^H(0, z))]^4.$$

Now let $\lambda_P(\zeta)|d\zeta|$ denote the Poincaré metric on $\text{Int}(P)$. It is well known (see for example [1]) that

$$\lambda_P(\zeta) \leq \frac{2}{d_{\mathbb{C}}(\zeta, \partial P)}.$$

Let $\tilde{q} = \zeta(z) \in P_h$. Then, if d_P^H denotes the Poincaré distance in $\text{Int}(P)$,

$$\begin{aligned} d_{\mathbb{D}}^H(0, z) &= d_P^H(\tilde{p}, \tilde{q}) \\ &= \inf \left\{ \int_{\gamma} \lambda_P(\zeta) |d\zeta|; \gamma \text{ is a path in } \text{Int}(P) \text{ from } \tilde{p} \text{ to } \tilde{q} \right\} \\ &\leq \inf \left\{ \int_{\gamma} \lambda_P(\zeta) |d\zeta|; \gamma \text{ is a path in } P_h \text{ from } \tilde{p} \text{ to } \tilde{q} \right\} \\ &\leq \frac{2d_{P_h}(\tilde{p}, \tilde{q})}{h} \\ &\leq \frac{2 \text{diam}_{P_h} P_h}{h}, \end{aligned}$$

which, together with (7), establishes the result. \square

Finally, the following *holomorphic removability* criterion will be needed: it is a consequence of Theorem V.3.2 on p. 202 of [12].

Theorem 27. *Let $\Omega \subset \mathbb{C}$ be a domain and $E \subset \Omega$ be a compact subset with finite 1-dimensional Hausdorff measure. Suppose $g: \Omega \rightarrow \mathbb{C}$ is an orientation-preserving homeomorphism onto its image which is conformal on $\Omega \setminus E$. Then, in fact, g is conformal on all of Ω .*

3. PAPER-FOLDING SCHEMES

This section provides the definition of paper-folding schemes, which are the main objects of study in this article. Some properties of the metric and measure on the scar of a paper-folding scheme are established (Lemma 33), and the notion of a *plain* paper-folding scheme is introduced.

The polygons used in the definition of paper-folding schemes are subsets of the Euclidean plane \mathbb{R}^2 , which will be identified with the complex plane \mathbb{C} in Sections 5 and 6 when the complex structure of paper surfaces is discussed. The notation $\hat{\mathbb{C}}$ will be used for the Riemann sphere.

Definitions 28 (Arc, segment, polygon, multipolygon). An *arc* in a metric space X is a homeomorphic image $\gamma \subset X$ of the interval $[0, 1]$. Its *endpoints* are the images of 0 and 1 and its *interior* is the image of $(0, 1)$, denoted $\overset{\circ}{\gamma}$. An *open arc* is the interior of an arc. A *segment* is an arc in \mathbb{R}^2

which is a subset of a straight line. The length of a segment α is denoted $|\alpha|$. A *simple closed curve* in X is a homeomorphic image of the unit circle.

An arc or simple closed curve in \mathbb{R}^2 is called *polygonal* if it is the concatenation of finitely many segments. Its *vertices* are the intersections of consecutive maximal segments, and the maximal segments themselves are its *edges*.

A *polygon* is a closed topological disk in \mathbb{R}^2 whose boundary is a polygonal simple closed curve. Its *vertices* are the same as its boundary's vertices and its *sides* are the edges forming its boundary.

A *multipolygon* is a disjoint union of finitely many polygons. A *(polygonal) multicurve* is a disjoint union of finitely many (polygonal) simple closed curves.

Definitions 29 (Segment pairing, interior pair, full collection, fold). Let $C \subset \mathbb{R}^2$ be an oriented polygonal multicurve and $\alpha, \alpha' \subset C$ be segments of the same length with disjoint interiors. The *segment pairing* $\langle \alpha, \alpha' \rangle$ is the relation which identifies pairs of points of α and α' in a length-preserving and orientation-reversing way. The segments α, α' and any two points which are identified under the pairing are said to be *paired*. Two paired points which lie in the interior of a segment pairing form an *interior pair*. Observe that the notation for a pairing is not ordered, so that $\langle \alpha, \alpha' \rangle$ and $\langle \alpha', \alpha \rangle$ represent the same pairing.

A collection $\{\langle \alpha_i, \alpha'_i \rangle\}$ of segment pairings is *interior disjoint* if the interiors of all of the segments α_i and α'_i are pairwise disjoint.

The *length* of a segment pairing $\langle \alpha, \alpha' \rangle$, denoted $|\langle \alpha, \alpha' \rangle|$, is the length of one of the arcs in the pairing, i.e., $|\langle \alpha, \alpha' \rangle| := |\alpha| = |\alpha'|$. If $\mathcal{P} = \{\langle \alpha_i, \alpha'_i \rangle\}$ is a (countable) interior disjoint collection of segment pairings on C , its *length*, denoted $|\mathcal{P}|$, is the sum of the lengths of the pairings in \mathcal{P} , i.e., $|\mathcal{P}| = \sum_i |\langle \alpha_i, \alpha'_i \rangle|$.

An interior disjoint collection \mathcal{P} of segment pairings on C is *full* if $|\mathcal{P}|$ equals half the length of C . This means that the pairings in \mathcal{P} cover C up to a set of Lebesgue 1-dimensional measure zero.

A pairing of two segments which have an endpoint in common is called a *fold* and the common endpoint is called its *folding point*. The folding point in a fold is therefore not paired with any other point.

An interior disjoint collection \mathcal{P} of segment pairings induces a reflexive and symmetric *pairing relation*, also denoted \mathcal{P} :

$$\mathcal{P} = \{(x, x') : x, x' \text{ are paired or } x = x'\}.$$

Definitions 30 (Paper-folding scheme). A *paper-folding scheme* is a pair (P, \mathcal{P}) where $P \subset \mathbb{R}^2$ is a multipolygon with the intrinsic metric d_P induced from \mathbb{R}^2 , and \mathcal{P} is a full interior disjoint collection of segment pairings on ∂P (positively oriented). The metric quotient $S := P/d_P^{\mathcal{P}}$ of P under the semi-metric $d_P^{\mathcal{P}}$ induced by the pairing relation \mathcal{P} is the associated *paper space*. When S is a closed (compact without boundary) topological surface, then (P, \mathcal{P}) is called a *surface paper-folding scheme* and S is the associated *paper surface*.

The projection map is denoted $\pi: P \rightarrow S$ and the quotient $G = \pi(\partial P) \subset S$ of the boundary is the *scar*. Notice that the restriction $\pi: \text{Int}(P) \rightarrow S \setminus G$ is a homeomorphism.

The (quotient) metric on S is denoted d_S . The metric d_G on G is defined to be the intrinsic metric as a subset of S : it will be shown in Lemma 33 below that this is equal to the quotient metric on G as a quotient of ∂P , where ∂P is endowed with its intrinsic metric as a subset of P .

The measure m_G on G is defined to be the push-forward of Lebesgue 1-dimensional measure $m_{\partial P}$ on ∂P . Hausdorff 1-dimensional measure on G is denoted μ_G^1 — it will be shown in Lemma 33 that $\mu_G^1 = \frac{1}{2}m_G$.

Example 35 below provides simple concrete examples of paper-folding schemes. The next definitions distinguish different types of points in a paper space.

Definitions 31 (Vertex, edge, singular point, planar point). For $k \in \mathbb{N} \cup \{\infty\}$, a point $x \in G$ is a *vertex of valence k* , or a *k -vertex*, if either (i) $\#\pi^{-1}(x) = k \neq 2$; or (ii) $\#\pi^{-1}(x) = k = 2$ and $\pi^{-1}(x)$ contains a vertex of P . Let \mathcal{V} denote the set of all vertices of G .

The points of the paper space S are divided into three types:

Singular points: vertices of valence ∞ and accumulations of vertices. Let \mathcal{V}^s denote the set of singular points.

Regular vertices: vertices which are not singular.

Planar points: all other points of S : that is, the points of $S \setminus \overline{\mathcal{V}}$.

The closures of the connected components of $G \setminus \overline{\mathcal{V}}$ are called *edges* of the scar G .

Remarks 32.

- i) When discussing paper foldings, symbols with tildes will usually refer to objects in the source P and symbols without tildes to objects in the quotient S . Thus, for example, \tilde{x} may denote a point in P and x its projection in S ; or $\langle \tilde{\alpha}, \tilde{\alpha}' \rangle$ may be a pairing on ∂P and $\alpha := \pi(\tilde{\alpha}) = \pi(\tilde{\alpha}') \subset G$. For notational simplicity, however, tildes will be omitted when all objects being discussed lie in P .
- ii) The metric on a multipolygon P is always the intrinsic metric induced by the metric on \mathbb{R}^2 , denoted d_P . The symbol $d_P^{\mathcal{P}}$ will be used to denote both the semi-metric on P induced by the pairing relation and the distance between $\sim_{\mathcal{P}}$ equivalence classes: if $\tilde{x}, \tilde{y} \in P$ and $[\tilde{x}], [\tilde{y}]$ denote their $\sim_{\mathcal{P}}$ -classes, then $d_P^{\mathcal{P}}([\tilde{x}], [\tilde{y}]) := d_P^{\mathcal{P}}(\tilde{x}, \tilde{y})$. Thus, if $x = \pi(\tilde{x}), y = \pi(\tilde{y}) \in S$, then $d_S(x, y) := d_P^{\mathcal{P}}([\tilde{x}], [\tilde{y}])$.
- iii) The endpoints of paired segments need not necessarily project to vertices or singular points. For example, a segment pairing $\langle \alpha, \alpha' \rangle$ can be split into two segment pairings $\langle \beta, \beta' \rangle, \langle \gamma, \gamma' \rangle$ by subdividing the segments α and α' , and the common endpoint of β and γ then projects to a planar point (see also Definition 39).

The next lemma summarises the properties of the metric and measure on the scar G which will be important later. One of the issues which it addresses is the equivalence of the two natural ways

to define a metric on G : as the intrinsic metric induced by the inclusion of G as a subset of (S, d_S) ; and as the quotient metric coming from the intrinsic metric on ∂P .

Lemma 33. *Let G be the scar of the paper-folding scheme (P, \mathcal{P}) . Then*

- a) *The set of planar points is open and dense in the scar G , while the set $\overline{\mathcal{V}}$ of vertices and singular points is a closed nowhere dense subset of G with zero m_G -measure.*
- b) *The intrinsic metric on G induced by the inclusion $G \subset S$ agrees with the quotient metric on $G = \partial P / d_{\partial P}^{\mathcal{P}}$ induced by the intrinsic metric $d_{\partial P}$ on ∂P .*
- c) *G has Hausdorff dimension 1, and Hausdorff 1-dimensional measure μ_G^1 on G is equal to $\frac{1}{2}m_G$.*
- d) *Every arc γ in G is rectifiable, and $|\gamma|_G = \frac{1}{2}m_G(\gamma)$.*

Proof.

- a) The set of planar points is open in G by definition (it is the set of points which are neither vertices nor accumulations of vertices), and is dense in G since it contains the image under the continuous surjection π of the (dense) set of points of ∂P which belong to interior pairs. The set of vertices and singular points is the complement of the set of planar points, and is therefore closed and nowhere dense.

$\pi^{-1}(\overline{\mathcal{V}})$ is contained in the complement of the set of points belonging to interior pairs, which has zero $m_{\partial P}$ -measure, so that $m_G(\overline{\mathcal{V}}) = 0$.

- b) Observe first that the intrinsic metrics d_P on P and $d_{\partial P}$ on ∂P satisfy $d_{\partial P}(\tilde{x}, \tilde{y}) \geq d_P(\tilde{x}, \tilde{y})$ for all $\tilde{x}, \tilde{y} \in \partial P$ (since any path in ∂P is also a path in P), and that $d_{\partial P}(\tilde{x}, \tilde{y}) = d_P(\tilde{x}, \tilde{y}) = d_{\mathbb{R}^2}(\tilde{x}, \tilde{y})$ if \tilde{x} and \tilde{y} lie in the same side of P .

Let d_G^i denote the intrinsic metric on G as a subset of (S, d_S) , and d_G^q denote the metric on G considered as the quotient metric space of $(\partial P, d_{\partial P})$ under the relation \mathcal{P} . It will be shown that $d_G^i \leq d_G^q$ and $d_G^q \leq d_G^i$, which will establish the result.

To show $d_G^i \leq d_G^q$: Let $x, y \in G$, and write $D = d_G^q(x, y)$. Let $\varepsilon > 0$ be any positive number. A path γ in G from x to y with length $|\gamma|_S < D + \varepsilon$ will be constructed, which will establish the result since $d_G^i(x, y) \leq |\gamma|_S$.

Let $\tilde{x}, \tilde{y} \in \partial P$ with $\pi(\tilde{x}) = x$ and $\pi(\tilde{y}) = y$. By definition of d_G^q , there is a \mathcal{P} -chain $((\tilde{p}_i, \tilde{q}_i))_{i=0}^k$ from \tilde{x} to \tilde{y} in ∂P with length

$$L^{\mathcal{P}}((\tilde{p}_i, \tilde{q}_i)) = \sum_{i=0}^k d_{\partial P}(\tilde{p}_i, \tilde{q}_i) < D + \varepsilon.$$

For each i , let $\tilde{\gamma}_i$ be a path from \tilde{p}_i to \tilde{q}_i in ∂P of length $d_{\partial P}(\tilde{p}_i, \tilde{q}_i)$ (the metric $d_{\partial P}$ is strictly intrinsic by Theorem 15), and let $\gamma_i = \pi \circ \tilde{\gamma}_i$. Since $\tilde{q}_i \mathcal{P} \tilde{p}_{i+1}$ for each i , the concatenation of the paths γ_i is a path γ in G from x to y ; and $|\gamma_i|_S \leq |\tilde{\gamma}_i|_P \leq |\tilde{\gamma}_i|_{\partial P} = d_{\partial P}(\tilde{p}_i, \tilde{q}_i)$ (the first inequality is from Remark 17, the second follows from $d_P \leq d_{\partial P}$, and the equality is by choice of the $\tilde{\gamma}_i$). Hence $|\gamma|_S < D + \varepsilon$ as required.

To show $d_G^q \leq d_G^i$: Let $x, y \in G$, and write $D = d_G^i(x, y)$. Let $\varepsilon > 0$ be any positive number. A \mathcal{P} -chain in ∂P which connects points \tilde{x} above x and \tilde{y} above y will be constructed with length less than $D + \varepsilon$, which will establish the result since $d_G^q(x, y)$ is less than or equal to the length of such a chain.

By definition of d_G^i there is a path $\gamma: [0, 1] \rightarrow G$ from x to y with $|\gamma|_S < D + \varepsilon$. That is,

$$\sum_{i=0}^{k-1} d_S(\gamma(t_i), \gamma(t_{i+1})) < D + \varepsilon$$

for any partition $0 = t_0 < t_1 < \dots < t_k = 1$.

Construct such a partition as follows. Let \tilde{e}_0 be a side of P which contains a point \tilde{x}_0 with $\pi(\tilde{x}_0) = \gamma(0) = x$, and let $t_1 \in [0, 1]$ be the greatest parameter for which there is a point $\tilde{x}_1 \in \tilde{e}_0$ with $\pi(\tilde{x}_1) = \gamma(t_1)$.

If $t_1 \neq 1$, then there must be a side $\tilde{e}_1 \neq \tilde{e}_0$ of P containing either \tilde{x}_1 or a point identified with it. Let $t_2 \in [t_1, 1]$ be the greatest parameter for which there is a point $\tilde{x}_2 \in \tilde{e}_1$ with $\pi(\tilde{x}_2) = \gamma(t_2)$. If $t_2 \neq 1$, then there is a side \tilde{e}_2 distinct from \tilde{e}_1 and \tilde{e}_0 containing \tilde{x}_2 or a point identified with it. Let $t_3 \in [t_2, 1]$ be the greatest parameter for which there is a point $\tilde{x}_3 \in \tilde{e}_2$ with $\pi(\tilde{x}_3) = \gamma(t_3)$.

Since P has only finitely many sides, this process terminates after a finite number of steps, yielding (after removing repeated parameters if necessary) a partition $0 = t_0 < t_1 < \dots < t_k = 1$ with the property that, for each i , there are points \tilde{p}_i and \tilde{q}_i above $\gamma(t_i)$ and $\gamma(t_{i+1})$ which lie on the same side of P . In particular, $d_{\partial P}(\tilde{p}_i, \tilde{q}_i) = d_S(\gamma(t_i), \gamma(t_{i+1}))$.

The sequence $((\tilde{p}_i, \tilde{q}_i))_{i=0}^{k-1}$ is therefore a \mathcal{P} -chain in ∂P with length

$$\sum_{i=0}^{k-1} d_{\partial P}(\tilde{p}_i, \tilde{q}_i) = \sum_{i=0}^{k-1} d_S(\gamma(t_i), \gamma(t_{i+1})) < D + \varepsilon$$

as required.

- c) Because $\pi: \partial P \rightarrow G$ is distance non-increasing, it follows from Lemma 20 that

$$\mu_G^1(\overline{\mathcal{V}}) \leq \mu_G^1(\pi(\overline{E})) \leq \mu_{\partial P}^1(\overline{E}) = m_{\partial P}(\overline{E}) = 0,$$

where $E \subset \partial P$ is the set of endpoints of segments in segment pairings. Since $m_G(\overline{\mathcal{V}}) = 0$ as well, it is enough to show that $\mu_G^1 = \frac{1}{2}m_G$ on $G \setminus \overline{\mathcal{V}}$. However this set has countably many connected components, each an open arc which is the homeomorphic and locally isometric image under π of exactly two disjoint open arcs in ∂P , and the result follows.

Since $\mu_G^1(G) > 0$ is finite, G has Hausdorff dimension 1 as required.

- d) Let γ be an arc in G , consider a partition of γ with points $x_0, \dots, x_n \in \gamma$, and let γ_i be the subarc of γ with endpoints x_i and x_{i+1} . Then, by Lemma 18,

$$\sum_{i=0}^{n-1} d_S(x_i, x_{i+1}) \leq \sum_{i=0}^{n-1} \text{diam}_G(\gamma_i) \leq \sum_{i=0}^{n-1} \mu_G^1(\gamma_i) \leq \mu_G^1(\gamma),$$

so that γ is rectifiable. That $|\gamma|_G = \frac{1}{2}m_G(\gamma)$ is immediate from c) and Lemma 19.

□

The main focus of this article is on paper-folding schemes whose quotient is a surface and, in particular, a sphere. For clarity of exposition, attention will be concentrated on *plain* folding schemes: these are both the most common and the simplest type of paper foldings and for them the paper space is always a sphere and the scar is always a dendrite (Theorem 42).

Definitions 34 (Unlinked pairing, plain paper-folding scheme). Let γ be a polygonal arc or polygonal simple closed curve.

Two pairs of (not necessarily distinct) points $\{x, x'\}$ and $\{y, y'\}$ of γ are *unlinked* if one pair is contained in the closure of a connected component of the complement of the other. Otherwise they are *linked*.

A symmetric and reflexive relation R on γ is *unlinked* if any two unrelated pairs of related points are unlinked: that is, if $x R x'$, $y R y'$, and neither x nor x' is related to either y or y' , then $\{x, x'\}$ and $\{y, y'\}$ are unlinked.

An interior disjoint collection \mathcal{P} of segment pairings on γ is *unlinked* if the corresponding relation \mathcal{P} is unlinked.

A paper-folding scheme (P, \mathcal{P}) is *plain* if P is a single polygon and \mathcal{P} is unlinked.

Example 35. Consider the unit square $P = \{(a, b) \in \mathbb{R}^2 : 0 \leq a, b \leq 1\}$ and pick a decreasing sequence of real numbers $(a_i)_{i \in \mathbb{N}}$ such that $\sum a_i = 1/2$. Define the following segment pairings on ∂P (Figures 1 and 2). The two vertical sides are paired and the top side is folded in half. On the bottom side define countably many folds of length a_i (i.e., pairings of two segments with one endpoint in common, each of length a_i), with pairwise disjoint interiors.

In Figure 1 the folds are placed in order of decreasing length from right to left with coincident endpoints. The complement of the folds is the bottom left vertex of P : all of the fold endpoints are identified with this vertex in the quotient space S .

In Figure 2, the folds are arranged in such a way that they are all disjoint. The closure of the complement of the folds is a Cantor set of Lebesgue measure 0, uncountably many of whose points are unpaired. This Cantor set is identified to a point in the quotient space S .

In both examples (and for any other way of arranging the folds) the paper-folding scheme is unlinked, and hence the quotient space is a topological sphere by Theorem 42 below. Notice also that the scars corresponding to the two schemes are isometric ∞ -ods (Definition 12).

These simple examples will be revisited in Example 43 below.

4. THE TOPOLOGICAL AND METRIC STRUCTURE OF PAPER SPACES

The first main theorem of this section, Theorem 42, states that the paper space associated to a plain paper folding is a sphere, and its scar is a dendrite. The tools used to prove this result are the dendrite quotient theorem (Theorem 7), which gives sufficient conditions for a quotient of a continuum to be a dendrite; and Moore's theorem (Theorem 5), which gives sufficient conditions for a quotient of a sphere to be a sphere.

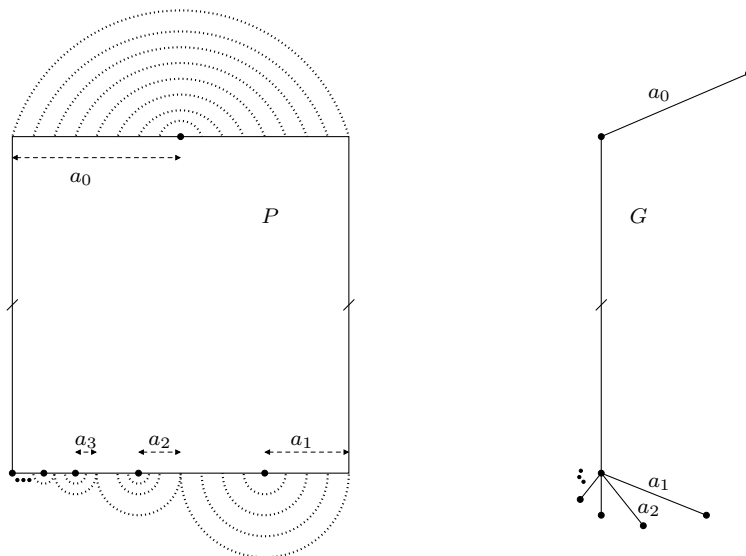


FIGURE 1. A paper-folding scheme and the associated scar.

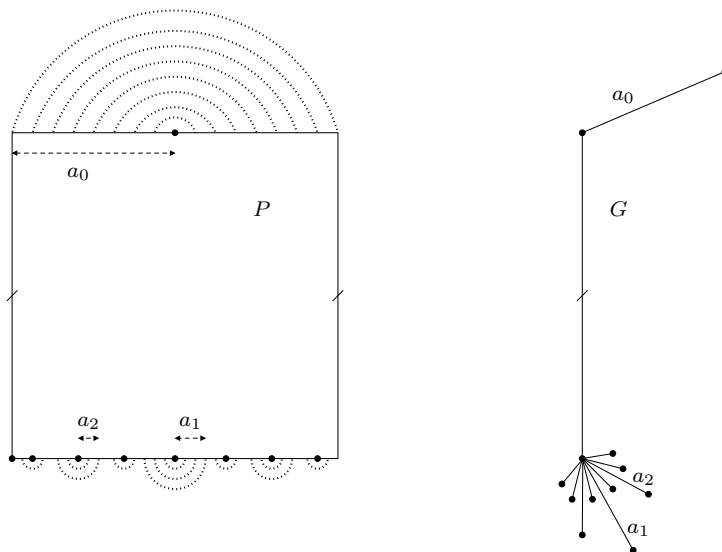


FIGURE 2. Another paper-folding scheme and the associated scar.

In Section 4.2 the topological structure of general paper spaces is considered: it is shown that they are paper surfaces provided that the boundary identifications are only finitely linked, and the structure of the scar in this case is described (Theorem 50). Finally, Section 4.3 contains a discussion of the metric structure of paper spaces.

4.1. The plain paper folding structure theorem. Many of the technical lemmas needed in the proof of the plain paper folding structure theorem are common to the general case and are not simplified by the plainness assumption. Thus in the beginning of this section (P, \mathcal{P}) is a general paper folding scheme: that is, \mathcal{P} is a full interior disjoint collection of segment pairings on the boundary ∂P of a multipolygon P ; the equivalence relation induced by the semi-metric $d_P^{\mathcal{P}}$ is denoted $\sim_{\mathcal{P}}$; the paper space is denoted S , the scar G , and the projection map $\pi: P \rightarrow S$; and if $x \in P$, its $\sim_{\mathcal{P}}$ -equivalence class is denoted $[x]$.

Definitions 36 (Plain arc, maximal plain arc). Let (P, \mathcal{P}) be a paper-folding scheme. An arc $\gamma \subset \partial P$ is *plain* if:

- a) Every pairing in \mathcal{P} which intersects the interior of γ is contained in γ (that is, if $\langle \alpha, \alpha' \rangle$ is a segment pairing and either α or α' intersects the interior of γ , then both α and α' are contained in γ); and
- b) The restriction of \mathcal{P} to γ is unlinked (Definitions 34).

A component γ of ∂P is *plain* if it is \mathcal{P} -saturated and the restriction of \mathcal{P} to γ is unlinked.

A plain arc is *maximal* if it is not strictly contained in any plain arc or component.

Remarks 37.

- a) Recall (Definitions 34) that a paper folding scheme (P, \mathcal{P}) is itself called plain if P is a single polygon and \mathcal{P} is unlinked. With the definitions just introduced this is equivalent to saying that P consists of a single polygon whose boundary ∂P is plain as just defined.
- b) Every plain arc contains at least one pairing: pairings are dense in ∂P and so intersect the interior of — and thus must be contained in — any plain arc or component.
- c) If every pairing which intersects the interior of γ is contained in γ , then the same holds for the complement of γ in ∂P : every pairing which intersects $\partial P \setminus \gamma$ is contained in $\overline{\partial P \setminus \gamma}$.
- d) The union of two plain arcs with non-empty intersection is plain. Hence any two distinct maximal plain arcs are disjoint.

Lemma 38. *If γ is a plain arc, then its endpoints are $\sim_{\mathcal{P}}$ -equivalent.*

Proof. Let a and b be the endpoints of γ : for convenience, they will be referred to as the left and right endpoints respectively, and the points of γ will be ordered from a to b .

Let $\varepsilon > 0$ be any positive number. It will be shown that $d_P^{\mathcal{P}}(a, b) < \varepsilon$, which will establish the result.

By fullness, there is a finite collection of segment pairings $\{\langle \alpha_r, \alpha'_r \rangle\}_{r=1}^N$ contained in γ which cover γ to within length ε . Label the segments so that α_r is to the left of α'_r for each r .

Let $p_0 = a$, and construct a \mathcal{P} -chain from a to b inductively as follows: for each $i \geq 0$, let q_i be the leftmost point of $[p_i, b]$ which is a left hand endpoint of some $\alpha_{r(i)}$, and let p_{i+1} be the right hand endpoint of $\alpha'_{r(i)}$, so that $q_i \mathcal{P} p_{i+1}$. Since the subcollection of segment pairings is finite, there is some k such that $[p_k, b]$ is disjoint from all of the α_r : set $q_k = b$. By construction, $p_0 \leq q_0 < p_1 \leq q_1 < \dots < p_k \leq q_k$.

The intervals (p_i, q_i) are disjoint from all of the segments α_r by choice of q_i . They are also disjoint from the α'_r . For if (p_i, q_i) were to contain some α'_r , then α_r couldn't lie in any (p_j, q_j) with $j < i$ as above; and if α_r were to lie in (q_j, p_{j+1}) then the segments $\alpha_{r(j)} < \alpha_r < \alpha'_{r(j)} < \alpha'_r$ would be linked, contradicting plainness.

Hence

$$d_{\mathcal{P}}^{\mathcal{P}}(a, b) \leq L^{\mathcal{P}}((p_i, q_i)) = \sum_{i=0}^k d(p_i, q_i) < \varepsilon$$

as required. □

Definition 39 (Subpairing). Let $\langle \alpha, \alpha' \rangle$ be a pairing in \mathcal{P} , β be a subarc of α , and β' be the corresponding subarc of α' . Then $\langle \beta, \beta' \rangle$ is said to be a *subpairing* of $\langle \alpha, \alpha' \rangle$.

Lemma 40. Let γ be a plain arc or component of ∂P , and $\langle \alpha, \alpha' \rangle$ be a pairing in \mathcal{P} , or a subpairing of a pairing in \mathcal{P} , which is contained in γ . Let $z \in \gamma \setminus (\overset{\circ}{\alpha} \cup \overset{\circ}{\alpha}')$, and if γ is an arc then assume that $\overset{\circ}{\alpha} \cup \overset{\circ}{\alpha}'$ separates z from the endpoints of γ .

Then the \mathcal{P} -semi-distance along ∂P from z to any point w in a different component of $\partial P \setminus (\overset{\circ}{\alpha} \cup \overset{\circ}{\alpha}')$ is bounded below by the length L of $\langle \alpha, \alpha' \rangle$:

$$d_{\partial P}^{\mathcal{P}}(z, w) \geq L := |\langle \alpha, \alpha' \rangle|.$$

Proof. Let γ_1 be the component of $\partial P \setminus (\overset{\circ}{\alpha} \cup \overset{\circ}{\alpha}')$ which contains z , and let γ_2 denote $\partial P \setminus (\gamma_1 \cup \overset{\circ}{\alpha} \cup \overset{\circ}{\alpha}')$. Since all of the pairings in \mathcal{P} are unlinked with $\langle \alpha, \alpha' \rangle$, both γ_1 and γ_2 are \mathcal{P} -saturated.

Let $((p_i, q_i))_{i=0}^k$ be a \mathcal{P} -chain in ∂P from z to w . By saturation of γ_1 , if $q_r \in \gamma_1$ then $p_{r+1} \in \gamma_1$: hence the last point of the chain which lies in γ_1 must be some p_r ; and by saturation of γ_2 , the first point of the chain after p_r which lies in γ_2 must be some q_s with $s \geq r$.

If $s = r$ then $d_{\partial P}(p_r, q_r) \geq L$. If $s > r$, then the points $q_r, p_{r+1}, q_{r+1}, \dots, p_s$ all lie in $\overset{\circ}{\alpha} \cup \overset{\circ}{\alpha}'$. Let $\xi, \xi' : [0, L] \rightarrow \partial P$ be parameterisations of α and α' by arc length, and let t_i be parameters such that q_i and p_{i+1} are equal to $\xi(t_i)$ or $\xi'(t_i)$ for $r \leq i < s$. Then

$$d_{\partial P}^{\mathcal{P}}(z, w) \geq \sum_{i=r}^s d_{\partial P}(p_r, q_r) \geq t_r + \sum_{i=r+1}^{s-1} |t_i - t_{i-1}| + |L - t_{s-1}| \geq L$$

as required. □

Recall that an arc γ is plain if the pairing relation \mathcal{P} has the properties that: every pairing in \mathcal{P} which intersects $\overset{\circ}{\gamma}$ is contained in γ ; and the restriction of \mathcal{P} to γ is unlinked. The following lemma states that analogous properties also hold for the equivalence relation $\sim_{\mathcal{P}}$.

Lemma 41. *Let γ be a plain arc and let a, b denote its endpoints. Then $\sim_{\mathcal{P}}$ is unlinked on γ and $\gamma \setminus [a] = \gamma \setminus [b]$ is $\sim_{\mathcal{P}}$ -saturated. Similarly, if γ is a plain component of ∂P , then $\sim_{\mathcal{P}}$ is unlinked on γ and γ is $\sim_{\mathcal{P}}$ -saturated.*

Proof. Unlinkedness: Assume that γ is a plain arc: the case where it is a plain component of ∂P is analogous. Let x, x', y , and y' be points of γ with $x \sim_{\mathcal{P}} x'$ and $y \sim_{\mathcal{P}} y'$, such that the pairs $\{x, x'\}$ and $\{y, y'\}$ are linked (and in particular all four points are distinct). It is necessary to show that all four points are $\sim_{\mathcal{P}}$ equivalent.

Without loss of generality, suppose that $a \leq x < y < x' < y' \leq b$ with respect to an order on γ . The three arcs $[x, y]$, $[y, x']$, and $[x', y']$ are all plain, by Lemma 40: for example, if a point of (x, y) were paired with a point of $[a, x]$ or a point of $(x', b]$, the segment pairing realizing this would separate x from x' , contradicting $x \sim_{\mathcal{P}} x'$; while if it were paired with a point of $(y, x']$ the segment pairing would separate y from y' . Hence $x \sim_{\mathcal{P}} y \sim_{\mathcal{P}} x' \sim_{\mathcal{P}} y'$ by Lemma 38.

Saturation: Let γ be a plain arc, and $x \in \overset{\circ}{\gamma}$. It is required to show that either $x \sim_{\mathcal{P}} a$, or $[x] \subset \gamma$.

If there is a pairing or subpairing in γ which separates x from a and b , then it separates x from $\partial P \setminus \gamma$, and $[x] \subset \gamma$ by Lemma 40.

If there is no such pairing, then the arcs $[a, x]$ and $[x, b]$ are both plain, and $[x] = [a] = [b]$ by Lemma 38.

If γ is a plain component of ∂P then it is clear that no point of γ can be $\sim_{\mathcal{P}}$ -equivalent to a point in another boundary component of P . \square

Theorem 42 (Topological structure of a plain paper folding). *The quotient S of a plain paper-folding scheme is a topological sphere, and its scar G is a dendrite.*

Proof. Since both ∂P and P are compact, the scar G and the paper space S are homeomorphic to the topological quotients $\partial P / \sim_{\mathcal{P}}$ and $P / \sim_{\mathcal{P}}$ respectively by Theorem 13.

To show that G is a dendrite, it is enough by Theorem 7 to show that the decomposition \mathcal{G} of ∂P into equivalence classes of $\sim_{\mathcal{P}}$ is dendritic. That \mathcal{G} is non-separated is immediate by Lemma 41: saying that $\sim_{\mathcal{P}}$ is unlinked is the same as saying that \mathcal{G} is non-separated. Thus it remains to show that if $[x] \in \mathcal{G}$ and $y \notin [x]$, then there is some $[z] \in \mathcal{G}$ which separates y and $[x]$.

Let $A \subset \partial P$ be the maximal arc containing y which is disjoint from $[x]$. Thus A is an open arc with endpoints $x_1, x_2 \in [x]$ (possibly $x_1 = x_2$ if $[x]$ is a single point). Since $\sim_{\mathcal{P}}$ is unlinked, A is $\sim_{\mathcal{P}}$ -saturated.

If there are paired points z_1 and z_2 in different components of $A \setminus \{y\}$, pick nearby interior paired points z'_1 and z'_2 , so that $[z'_1] = [z'_2]$ does not contain y . Then $[z'_1]$ separates y and $[x]$ as required.

If there are no such points then $[x_1, y]$ and $[y, x_2]$ are plain, so that $y \in [x]$ by Lemma 38, a contradiction.

To show that the paper space S is a sphere, it is convenient to change coordinates by a suitable homeomorphism so that ∂P is the unit circle S^1 in $\widehat{\mathbb{C}}$, and P is the exterior of the unit disk $\mathbb{D} \subset \widehat{\mathbb{C}}$:

the unit disk \mathbb{D} will be regarded as the hyperbolic plane. This change of coordinates is purely for convenience: the constructions don't use hyperbolic geometry in any essential way.

Connect the points x, y of each interior pair in ∂P with the geodesic having them as endpoints. Because \mathcal{P} is unlinked, these geodesics are pairwise disjoint. Let \mathcal{L} be the geodesic lamination obtained by taking the closure of the union of such geodesics. Because the equivalence relation $\sim_{\mathcal{P}}$ is closed, every geodesic in \mathcal{L} joins a pair of equivalent points.

If U is a component of $\mathbb{D} \setminus \mathcal{L}$, let \overline{U} denote its closure in S . Then \overline{U} is a closed disk, whose boundary is the union of some geodesics (including endpoints) in \mathcal{L} joining equivalent points which are not in interior pairs, and other points of ∂P .

Let $x, y \in \partial P$ be distinct points which are not in interior pairs, and let $[x, y] \subset \partial P$ be an interval with these endpoints. Then the following conditions are equivalent:

- i) $x \sim_{\mathcal{P}} y$.
- ii) $[x, y]$ is plain.
- iii) x and y lie in the closure \overline{U} of the same component U of $\mathbb{D} \setminus \mathcal{L}$ or are the endpoints of a geodesic in \mathcal{L} .

For **i) \iff ii)**, suppose that $[x, y]$ is not plain. Since \mathcal{P} is unlinked, this means that there is a pairing $\langle \alpha, \alpha' \rangle$ which intersects (x, y) but is not contained in $[x, y]$. The assumption that x and y are not in interior pairs means that one of the segments, say α , is contained in $[x, y]$ while the other, α' , is disjoint from (x, y) . Thus x and y lie in different components of $\partial P \setminus (\overset{\circ}{\alpha} \cup \overset{\circ}{\alpha}')$, so that $x \not\sim_{\mathcal{P}} y$ by Lemma 40. The converse is immediate from Lemma 38.

For **ii) \iff iii)**: Suppose that $[x, y]$ is plain and that the geodesic with endpoints x and y is not in \mathcal{L} . Then there are no geodesics in \mathcal{L} with one endpoint in (x, y) and the other disjoint from $[x, y]$, and hence the geodesic with endpoints x and y lies in a single component U of $\mathbb{D} \setminus \mathcal{L}$, so that $x, y \in \overline{U}$. If $[x, y]$ is not plain then as above there is a pairing $\langle \alpha, \alpha' \rangle$ with $\overset{\circ}{\alpha} \subset (x, y)$ and $\overset{\circ}{\alpha}' \cap [x, y] = \emptyset$, and any geodesic connecting paired points in $\overset{\circ}{\alpha}$ and $\overset{\circ}{\alpha}'$, together with its endpoints, separates x from y in $\overline{\mathbb{D}}$.

Therefore two distinct points $x, y \in \partial P$ satisfy $x \sim_{\mathcal{P}} y$ if and only if either they are the endpoints of a geodesic in \mathcal{L} , or they lie in the closure of the same component of $\mathbb{D} \setminus \mathcal{L}$, so that a decomposition \mathcal{G} of $\widehat{\mathbb{C}}$ which realizes $\sim_{\mathcal{P}}$ can be constructed with the following elements:

- I) Closures \overline{U} of components of $\mathbb{D} \setminus \mathcal{L}$;
- II) Geodesics in \mathcal{L} , together with endpoints, which are not contained in elements of type I); and
- III) Points which are not contained in elements of types I) and II).

The elements of \mathcal{G} are closed disks, arcs, and points, and so are connected and do not separate $\widehat{\mathbb{C}}$. To complete the proof using Moore's theorem, it is therefore only required to show that \mathcal{G} is upper semi-continuous. So let $x_n \rightarrow x$ and $y_n \rightarrow y$ be sequences in $\widehat{\mathbb{C}}$ such that x_n and y_n belong to

the same decomposition element for all n : it is required to show that x and y belong to the same decomposition element.

If x_n and y_n belong to elements of type III) for infinitely many n , then $x_n = y_n$ for these values of n , and hence $x = y$. Passing to a subsequence, it can therefore be assumed that none of the x_n or y_n belong to an element of type III).

Suppose that infinitely many of the x_n and y_n belong to geodesics γ_n of type II). If there is no positive lower bound on the distance in ∂P between the endpoints of the γ_n then $x = y$. If there is such a lower bound then the geodesics γ_n converge to a geodesic (together with endpoints) γ in \mathcal{L} which contains x and y .

It can therefore be assumed that all of the x_n and y_n belong to elements \overline{U}_n of type I). For each n , let $\alpha_n \subset \overline{U}_n$ be the geodesic arc with endpoints x_n and y_n , and let γ_n be the geodesic containing α_n . If there is no lower bound on the distance in ∂P between the endpoints of the γ_n , then $x = y$. If there is such a lower bound then the geodesics γ_n converge to a geodesic γ containing x and y .

If either $x = y$ or γ is a geodesic in \mathcal{L} , then x and y lie in the same decomposition element. Otherwise, the geodesic arc $\alpha \subset \gamma$ connecting x and y is disjoint from \mathcal{L} (if it intersected a geodesic in \mathcal{L} , then so would α_n for large n), and hence x and y lie in the same component of $\mathbb{D} \setminus \mathcal{L}$. \square

Example 43. Here the construction in the above proof is considered for the paper-folding schemes of Example 35. Recall that P is a square, the top side of P is folded in half, the vertical sides are paired, and folds of lengths a_i are placed along the bottom side, either contiguously from right to left (Figure 1) or disjointly (Figure 2).

Figure 3 depicts the decomposition of $\widehat{\mathbb{C}}$ which realizes $\sim_{\mathcal{P}}$ in the scheme of Figure 1. There is a single decomposition element of type I) which is denoted g_∞ : it includes countably many points of ∂P (the fold endpoints and the bottom left corner of P), and countably many geodesics joining these points in pairs. The other geodesics (including their endpoints on ∂P) are decomposition elements of type II), and the points of $\text{Int}(P)$ are decomposition elements of type III).

The decomposition for the scheme of Figure 2 is similar: in this case the single decomposition element g_∞ of type I) intersects ∂P in a Cantor set, only countably many of whose points are connected by geodesics.

4.2. Surface paper folding schemes. In this section the topological structure of the paper space S of a general paper-folding scheme (P, \mathcal{P}) is considered. It is clear that linked identifications around ∂P create handles in the paper space, and that if there are infinitely many such handles then S cannot be a compact surface. The main result of this section, Theorem 50, states that this is the only obstruction to S being a paper surface, and describes the scar G in the surface case: each component of G (corresponding to a component of S) is the union of countably many dendrites (one corresponding to each maximal plain arc in ∂P) and a finite connected graph (corresponding to the linked identifications), each dendrite being attached to the connected graph at a single point.

The first step is to reduce to the case in which P is a single polygon: or, more accurately, a single closed disk, since the polygonal structure of P will be lost in the constructions which follow.

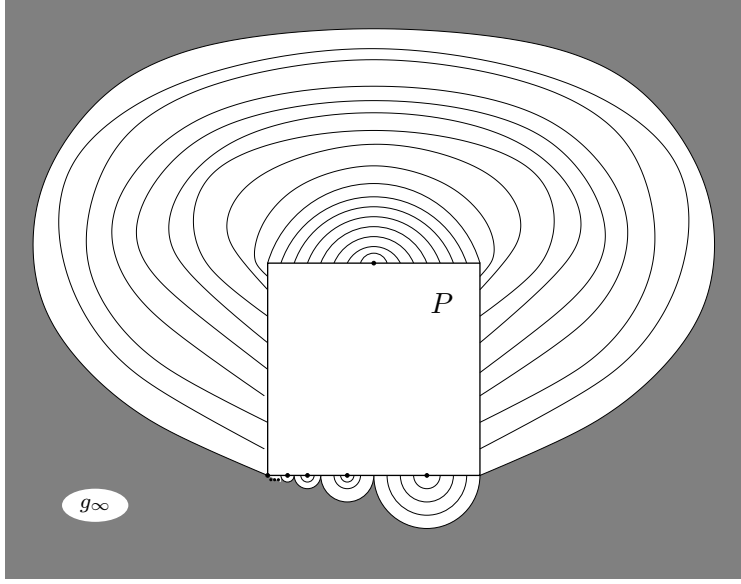


FIGURE 3. The musc decomposition realising $\sim_{\mathcal{P}}$ in the paper-folding scheme of Figure 1.

Let Υ be the finite graph whose vertices are the components P_i of P , and which has an edge joining P_i and P_j if and only if there is a pairing $\langle \alpha, \alpha' \rangle \in \mathcal{P}$ with $\alpha \subset \partial P_i$ and $\alpha' \subset \partial P_j$. Then the connected components of S correspond bijectively to the connected components of Υ . In the remainder of the section it will be assumed that Υ is connected (if not, apply the results to each connected component in turn).

Let T be a spanning subtree of Υ . Pick one pairing corresponding to each edge of T , and realize the associated identifications. The resulting space is a topological disk P' , and the remaining pairings induce a full interior disjoint collection \mathcal{P}' of arc pairings along $\partial P'$. It therefore suffices to consider (P', \mathcal{P}') . As mentioned above, P' is not a polygon, but the polygonal structure is irrelevant to the topological structure of the quotient: the word “segment” will be understood to mean “arc”, and the length of an arc in $\partial P'$ is given by the minimal length of a subset of ∂P which projects to it.

In the remainder of this section, then, it will be assumed that P is a topological disk with rectifiable boundary, and that \mathcal{P} is a full interior disjoint collection of arc pairings on ∂P . It can also be assumed that (P, \mathcal{P}) is not plain, since otherwise the results of Section 4.1 apply.

Let $\Pi = \{\gamma_i : i \in I\}$ be the set of maximal plain arcs in ∂P , where I is a finite or countable index set. These arcs are mutually disjoint (Remarks 37d)), and each component of $\partial P \setminus \bigcup_{i \in I} \gamma_i$ is an open arc (between any two elements of Π lies an arc α of a pairing of \mathcal{P} whose interior is not contained in any plain arc).

Let \mathcal{C} be the simple closed curve obtained from ∂P by collapsing each element of Π to a point (in fact the maximal plain arcs will not be collapsed: rather the identifications on them will be

realised, and in particular their endpoints will be identified). The pairings in \mathcal{P} which are not contained in any element of Π induce a full interior disjoint collection \mathcal{Q} of arc pairings on \mathcal{C} , with respect to which \mathcal{C} contains no plain arcs, since the preimage in ∂P of such a plain arc would be a plain arc intersecting the complement of the γ_i .

The aim now is to decompose \mathcal{C} into maximal *unlinked* arcs.

Definitions 44 (Unlinked arc, maximal unlinked arc). An arc γ in \mathcal{C} is *unlinked* if

- a) Every arc β_i in a pairing in \mathcal{Q} which intersects $\overset{\circ}{\gamma}$ is contained in γ ;
- b) $\overline{\bigcup_i \beta'_i}$ is an arc γ' , where the β'_i are the arcs paired with the arcs β_i of a); and
- c) The restriction of \mathcal{Q} to $\gamma \cup \gamma'$ is unlinked on \mathcal{C} .

The arc γ' is said to be *paired* with γ . An unlinked arc γ is *maximal* if it is not strictly contained in any other unlinked arc.

Remarks 45.

- a) An unlinked arc γ is disjoint from its paired arc γ' , as otherwise $\gamma \cup \gamma'$ would be a plain arc.
- b) Let γ be an unlinked arc with pair γ' . Then the restriction of $\sim_{\mathcal{Q}}$ to $\gamma \cup \gamma'$ is the arc pairing $\langle \gamma, \gamma' \rangle$.

Lemma 46. \mathcal{C} can be written uniquely as a union of maximal unlinked arcs and points which are not contained in any unlinked arc. The maximal unlinked arcs intersect only at their endpoints.

Proof. If γ_1 and γ_2 are unlinked arcs whose interiors intersect, then $\gamma_1 \cup \gamma_2$ is also an unlinked arc. In particular, two distinct maximal unlinked arcs can only intersect at their endpoints.

Let β be an arc in a pairing in \mathcal{Q} . Then β is itself an unlinked arc. Let γ_β be the union of all unlinked arcs containing β . Then γ_β is either an arc, or an arc without one or both of its endpoints, which satisfies the conditions of Definition 44 except for the possible absence of endpoints of γ_β and γ'_β . However, this means that the conditions also hold for γ_β together with its endpoints, so that γ_β is in fact an arc.

Hence every arc in a pairing in \mathcal{Q} is contained in a unique maximal unlinked arc, and these arcs intersect only at their endpoints. If x is a point of \mathcal{C} which is not contained in any of these arcs, then any arc containing x must intersect a paired arc β and hence the maximal arc γ_β : therefore x is not contained in any unlinked arc. \square

Definition 47 (Finitely linked). (P, \mathcal{P}) is said to be *finitely linked* if there are only finitely many maximal unlinked arcs in the decomposition of Lemma 46.

Remark 48. If (P, \mathcal{P}) is finitely linked, then there cannot be any points of \mathcal{C} which are disjoint from all of the maximal unlinked arcs: that is, \mathcal{C} can be written uniquely as a union of finitely many maximal unlinked arcs.

The following lemma is the main tool used in the proof of the converse of Theorem 50 below, that if (P, \mathcal{P}) is not finitely linked then S is not a compact surface. The setting of the lemma is slightly different from that of the remainder of the paper — P is a compact surface with a single

boundary component \mathcal{C} rather than a topological disk — but the concepts of *arc pairings on \mathcal{C}* , *plain arcs*, and *finitely linked* are defined analogously.

Lemma 49. *Let P be a compact surface with a single boundary component \mathcal{C} , and \mathcal{P} be a full interior disjoint collection of arc pairings on \mathcal{C} which is such that \mathcal{C} contains no plain arcs, and is not finitely linked. Then there are maximal unlinked arcs α and β , with paired arcs α' and β' , such that $\langle \alpha, \alpha' \rangle$ and $\langle \beta, \beta' \rangle$ are linked; and carrying out the identifications corresponding to these arc pairings yields a compact surface \hat{P} having a single boundary component $\hat{\mathcal{C}}$, with the properties that:*

- *the genus of \hat{P} is one greater than that of P ;*
- *the induced arc pairings $\hat{\mathcal{P}}$ on $\hat{\mathcal{C}}$ are such that $\hat{\mathcal{C}}$ contains no plain arcs and \hat{P} is not finitely linked.*

Proof. First some notation will be introduced. Let \mathcal{A} be the set of maximal unlinked arcs on \mathcal{C} , and $\phi: \mathcal{A} \rightarrow \mathcal{A}$ be the involution which takes each maximal unlinked arc to its pair. Fix an orientation of \mathcal{C} , and for each pair of distinct points $x, y \in \mathcal{C}$, write $[x, y]$ for the arc in \mathcal{C} with initial and final endpoints x and y .

A *segment* in \mathcal{A} is a set of the form

$$[[x, y]] := \{\gamma \in \mathcal{A} : \gamma \subset [x, y]\},$$

where x and y are distinct points of \mathcal{C} which do not lie in the interior of any element of \mathcal{A} . A segment is *non-trivial* if it has more than one element. Notice that each segment is endowed with a total order $<$ induced by the orientation on \mathcal{C} . Similarly, given distinct elements γ and δ of \mathcal{A} , write $[[\gamma, \delta]]$ for the segment $[[x, y]]$, where x is the initial point of γ and y is the final point of δ , provided that these points do not coincide; and write $[\gamma, \delta]$ for the arc $[x, y]$ in \mathcal{C} .

The set \mathcal{A} has the following properties:

- i) (The elements of \mathcal{A} are maximal.) There are no two non-trivial disjoint segments $[[x, y]]$ and $[[x', y']]$ with the properties that: $\gamma \in [[x, y]]$ if and only if $\phi(\gamma) \in [[x', y']]$; and $\gamma < \delta$ in $[[x, y]]$ if and only if $\phi(\gamma) > \phi(\delta)$ in $[[x', y']]$. (If there were such segments, then arcs in each segment could be coalesced to give larger unlinked arcs.)
- ii) (There are no plain arcs.) Let $\gamma \in \mathcal{A}$, and suppose that $\phi(\delta) \in [[\gamma, \phi(\gamma)]]$ whenever $\delta \in [[\gamma, \phi(\gamma)]]$. Then there are $\alpha, \beta \in [[\gamma, \phi(\gamma)]]$ such that $\alpha < \beta < \phi(\alpha) < \phi(\beta)$ with respect to the order on $[[\gamma, \phi(\gamma)]]$. (Otherwise $[\gamma, \phi(\gamma)]$ would be a plain arc.)

The existence of the maximal unlinked arcs α and β in the statement of the lemma is immediate from property ii). Let γ be any element of \mathcal{A} . If there is some $\delta \in \mathcal{A}$ with $\delta \in [[\gamma, \phi(\gamma)]]$ but $\phi(\delta) \notin [[\gamma, \phi(\gamma)]]$, then set $\alpha = \gamma$ and $\beta = \delta$. If there is no such δ , then suitable arcs α and β exist by property ii).

It will be assumed in the remainder of the proof that the arcs α , α' , β , and β' have no common endpoints: only minor modifications are required in the case where they share some endpoints.

Carrying out the identifications corresponding to these arcs clearly yields a compact surface \hat{P} with genus one greater than that of P , having a single boundary component $\hat{\mathcal{C}}$. Let $\pi: \mathcal{C} \setminus \text{Int}(\alpha \cup$

$\alpha' \cup \beta \cup \beta') \rightarrow \hat{\mathcal{C}}$ be the natural projection: notice that π does not respect the cyclic orders around \mathcal{C} and $\hat{\mathcal{C}}$. The image under π of the endpoints of the identified arcs consists of four points $p_1, p_2, p_3, p_4 \in \hat{\mathcal{C}}$. Denote by $\hat{\mathcal{C}}_1, \hat{\mathcal{C}}_2, \hat{\mathcal{C}}_3$, and $\hat{\mathcal{C}}_4$ the four components of $\hat{\mathcal{C}} \setminus \{p_1, p_2, p_3, p_4\}$; and let $\hat{\mathcal{A}} = \mathcal{A} \setminus \{\alpha, \beta, \alpha', \beta'\}$. Notice that the $\hat{\mathcal{C}}_i$ can be regarded as segments in both \mathcal{C} and $\hat{\mathcal{C}}$, and that each is totally ordered by the orientation of \mathcal{C} (and consistently by the induced orientation of $\hat{\mathcal{C}}$). If the $\hat{\mathcal{C}}_i$ are labelled so that they appear in the order $\hat{\mathcal{C}}_1, \hat{\mathcal{C}}_2, \hat{\mathcal{C}}_3, \hat{\mathcal{C}}_4$ around \mathcal{C} , then they appear in the reverse order $\hat{\mathcal{C}}_4, \hat{\mathcal{C}}_3, \hat{\mathcal{C}}_2, \hat{\mathcal{C}}_1$ around $\hat{\mathcal{C}}$.

If $\gamma \in \hat{\mathcal{A}}$ then $\pi(\gamma)$ is an unlinked arc, with pair $\pi(\phi(\gamma))$, but it need not be maximal. However, it will be shown that there are only finitely many $\gamma \in \hat{\mathcal{A}}$ for which $\pi(\gamma)$ is not maximal, which will establish that $\hat{\mathcal{P}}$ is not finitely linked, as required.

To prove this, suppose that Γ is a subset of $\hat{\mathcal{A}}$ such that $\overline{\bigcup_{\gamma \in \Gamma} \pi(\gamma)}$ is an unlinked arc. Suppose that Γ contains two arcs γ and δ such that $\pi(\gamma) < \pi(\delta)$ lie in a single $\hat{\mathcal{C}}_i$, and $\pi(\phi(\gamma))$ and $\pi(\phi(\delta))$ also lie in a single $\hat{\mathcal{C}}_j$. Then the segment $[\gamma, \delta]$ would be unlinked, contradicting the assumption that the elements of \mathcal{A} are maximal unlinked arcs. Hence Γ contains at most three arcs, which are adjacent, and there is some $\gamma \in \Gamma$ such that either $\pi(\gamma)$ or $\pi(\phi(\gamma))$ contains one of the points p_i . There are therefore at most 24 elements γ of $\hat{\mathcal{A}}$ for which $\pi(\gamma)$ is not maximal, as required.

It remains to show that $\hat{\mathcal{C}}$ contains no plain arcs. Suppose for a contradiction that it does contain some plain arc δ . Let

$$\Delta = \{\gamma \in \hat{\mathcal{A}} : \pi(\gamma) \subset \delta\},$$

and observe that $\phi(\Delta) = \Delta$ since δ is saturated. Moreover, if $\gamma \in \Delta$ then $\pi(\gamma)$ and $\pi(\phi(\gamma))$ lie in different components $\hat{\mathcal{C}}_i$, as otherwise either $[\gamma, \phi(\gamma)]$ or $[\phi(\gamma), \gamma]$ would be a plain arc in \mathcal{C} . In particular, δ contains at least one of the points p_i in its interior.

If Δ is infinite, then pick an infinite segment $[[x, y]] \subset \Delta$ contained in a single $\hat{\mathcal{C}}_i$. Since δ is plain, the arcs $\{\phi(\gamma) : \gamma \in [[x, y]]\}$ lie in the union of the other three components in exactly the opposite order to their pairs in $[[x, y]]$. Picking a subsegment so that the $\phi(\gamma)$ lie in a single component gives a contradiction to property i) above.

On the other hand, if Δ is finite then it contains an arc γ such that $\pi(\gamma)$ and $\pi(\phi(\gamma))$ share a common endpoint, which must be one of the points p_i . Suppose, without loss of generality, that the common endpoint is p_1 , which is the image under π of the initial point of α and the final point of $\phi(\alpha)$. If the endpoint of γ (respectively $\phi(\gamma)$) coincides with the initial point of α then the segment $[[\gamma, \alpha]]$ (respectively $[[\phi(\gamma), \alpha]]$) gives a contradiction to property i) above. \square

Theorem 50. *A connected paper-folding scheme (P, \mathcal{P}) is a surface paper folding scheme if and only if it is finitely linked. In this case, the scar G is a local dendrite, which can be written as $G = C \cup \Gamma$, where*

- *C is a finite connected graph in S with the property that any simple closed curve in C is homotopically non-trivial; and*

- Γ is a union of finitely or countably many disjoint dendrites, with diameters decreasing to 0, each of which intersects C exactly once.

Proof. Suppose first that (P, \mathcal{P}) is finitely linked. The identifications $\sim_{\mathcal{P}}$ on ∂P will be done in two steps. First those arising from pairings contained in plain arcs are carried out: applying Moore's theorem as in the proof of Theorem 42, these yield a topological sphere on which the simple closed curve \mathcal{C} bounds a disk whose interior contains only points of $\widehat{\mathbb{C}} \setminus P$. Removing this disk and carrying out the remaining (finitely many) identifications along \mathcal{C} yields a topological surface. The details are similar to those of the proof of Theorem 42, and will only be sketched.

As in that proof, regard ∂P as the unit circle in $\widehat{\mathbb{C}}$, so that P is the exterior of the unit disk $\overline{\mathbb{D}}$, which is regarded as the hyperbolic plane. Connect the points of each interior pair in ∂P which is contained in a plain arc with the geodesic joining them, and include also the geodesic g_{γ} joining the endpoints of each maximal plain arc γ . Let \mathcal{L} be the geodesic lamination obtained by taking the closure of the union of these geodesics. For each maximal plain arc γ , let D_{γ} denote the disk bounded by $\gamma \cup g_{\gamma}$.

The musc decomposition \mathcal{G} of $\widehat{\mathbb{C}}$ with the following elements then realizes the equivalence relation \sim_{Plain} on ∂P corresponding to the set of pairings contained in plain arcs:

- I) Closures of components of $D_{\gamma} \setminus \mathcal{L}$, for each maximal plain arc γ ;
- II) Geodesics in \mathcal{L} (together with endpoints) which are not contained in elements of type I); and
- III) Points which are not contained in elements of types I) and II).

The quotient of $\widehat{\mathbb{C}}$ by \sim_{Plain} is therefore a topological sphere containing the simple closed curve \mathcal{C} obtained from ∂P by collapsing each maximal plain arc to a point. One complementary component of \mathcal{C} is (the projection of) $\widehat{\mathbb{C}} \setminus \left(P \cup \bigcup_{\gamma \text{ plain}} D_{\gamma} \right)$, while the other contains (the projection of) the interior of P , together with a dendrite corresponding to each maximal plain arc γ , attached to \mathcal{C} at a single point. Removing the former complementary component yields a topological disk bounded by \mathcal{C} , and carrying out the (finitely many) identifications on \mathcal{C} gives the compact surface S : the bounding curve \mathcal{C} projects to a finite graph C in S in which every simple closed curve is homotopically non-trivial, and the scar G consists of this graph together with the dendrites arising from maximal plain arcs.

For the converse, suppose that (P, \mathcal{P}) is not finitely linked. Carry out the identifications realizing \sim_{Plain} as above, and remove the complementary component of \mathcal{C} corresponding to the exterior of P . Using Lemma 49, it is possible to carry out repeated additional pairs of identifications, each of which adds 1 to the genus of the resulting surface. It follows that S contains a surface (with boundary) of genus n for all n , so that S cannot be a compact surface as required. \square

The injectivity radius of G , which gives the length of the shortest simple closed curve in the scar, will be important in Section 5.

Definition 51 (Injectivity radius of G). The *injectivity radius* \bar{r} of G is defined by

$$\bar{r} = \frac{1}{2} \inf\{|\gamma|_G : \gamma \text{ is a simple closed curve contained in } G\}.$$

Lemma 52. *Let G be the scar of a finitely linked paper-folding scheme. Then the injectivity radius of G satisfies $\bar{r} > 0$. For every $x \in G$ and every $0 \leq r < \bar{r}$, the closed ball $\overline{B}_G(x; r)$ is a dendrite and is thus contractible.*

Proof. $\bar{r} > 0$ since the graph C of Theorem 50 is finite, and hence there are only finitely many simple closed curves in G .

Let $r < \bar{r}$. Then $\overline{B}_G(x; r)$ is a continuum (it is connected since the metric d_G is intrinsic) which contains no simple closed curves. Each dendrite in the decomposition of G intersects $\overline{B}_G(x; r)$ in a subcontinuum, and hence in a dendrite (Theorem 8e)); and similarly C intersects $\overline{B}_G(x; r)$ in a tree. Hence any two distinct points of $\overline{B}_G(x; r)$ are separated by a third point of $\overline{B}_G(x; r)$, establishing by Theorem 8a) that it is a dendrite as required. \square

4.3. The metric structure of paper spaces. Let S be a paper space. Near planar points, S is Euclidean: that is, such points have an open neighborhood which is isometric to an open ball in \mathbb{R}^2 . Similarly, near a regular vertex x , S is isometric to the apex of a cone, with cone angle equal to the sum of the internal angles of the multipolygon P at the points of $\pi^{-1}(x)$. The following definitions formalize this statement. Throughout this section (P, \mathcal{P}) is an arbitrary paper-folding scheme with scar G and associated paper space S .

Definition 53 (Cone angle at non-singular points). Let $x \in G$ be non-singular with $\pi^{-1}(x) = \{\tilde{x}_1, \dots, \tilde{x}_k\}$, and let $\tilde{\eta}_i$ be the internal angle of P at \tilde{x}_i . The *cone angle* at x is the number $\eta(x) = \tilde{\eta}_1 + \dots + \tilde{\eta}_k$.

In particular, the cone angle at a regular point $x \in G$ is $\eta(x) = 2\pi$.

Definition 54 (Metric cone: see [3]). Let X be a topological space. The *cone* $\text{Cone}(X)$ over X is the (topological) quotient of $[0, \infty) \times X$ by the equivalence relation which collapses $\{0\} \times X$ to a point, that is, whose only non-trivial class is $\{0\} \times X$. This point in the quotient is the *origin* or *apex* of the cone and is also denoted 0. If (X, d) is a metric space, then it is possible to make $\text{Cone}(X)$ into a metric space by defining a distance as follows: if $p = [t, x], q = [s, y] \in \text{Cone}(X)$, set

$$d_c(p, q) = \begin{cases} \sqrt{t^2 + s^2 - 2st \cos(d(x, y))}, & \text{if } d(x, y) \leq \pi \\ t + s, & \text{if } d(x, y) \geq \pi. \end{cases}$$

Let S_r^1 denote the circle of radius $r > 0$ in \mathbb{R}^2 with the intrinsic metric. Then the cone $\text{Cone}(S_r^1)$ is locally isometric to the plane itself, except at the apex for $r \neq 1$, and $\text{Cone}(S_1^1)$ is globally isometric to \mathbb{R}^2 . Moreover, two cones on circles $\text{Cone}(S_{r_1}^1), \text{Cone}(S_{r_2}^1)$ are globally isometric only if $r_1 = r_2$.

Definition 55 (Conic-flat surface). A *conic-flat surface* is a metric space which is locally isometric to cones on circles: for every point x , there exist $r, \varepsilon > 0$ and $c \in \text{Cone}(S_r^1)$ such that $B(x; \varepsilon)$ is isometric to $B_{\text{Cone}(S_r^1)}(c; \varepsilon)$.

Theorem 56 (Metric structure of paper spaces).

- a) Let x be a non-singular point of G . Then there exists $r > 0$ such that $B_S(x; r)$ is isometric to $B_{\text{Cone}(S_{\eta(x)/2\pi}^1)}(0; r)$. In particular, $S \setminus \mathcal{V}^s$ is a conic-flat surface.
- b) The metric d_S on a paper space S is strictly intrinsic.

Proof. The proof of Part a) is straightforward and technical and is omitted: this statement is not used in the remainder of the paper. Part b) is immediate from Lemma 16 and Theorem 15 since the metric d_P on P is intrinsic, and P (and hence S) is compact. \square

Notice that even if S is a paper surface, the metric on the conic-flat surface $S \setminus \mathcal{V}^s$ need not be strictly intrinsic.

5. THE CONFORMAL STRUCTURE ON PAPER SURFACES

If (P, \mathcal{P}) is a surface paper-folding scheme and S is the quotient paper surface, then there is a natural conformal structure on the set $\text{Planar}(S)$ of planar points coming from the local Euclidean structure. The question addressed in this section is whether or not this complex structure extends uniquely across non-planar points of S .

The complex structure extends readily across regular vertices of G using the conic-flat structure on $S \setminus \mathcal{V}^s$ described in Section 4.3: at a cone point of angle η , the map $z \mapsto z^{2\pi/\eta}$ can be used to introduce conformal coordinates. Thus the case of interest is that of isolated singular points. Theorem 59 below provides a criterion for the complex structure to extend uniquely across such a point. In particular, if all singular points are isolated, and this criterion holds at each of them, then S is a Riemann surface. Similar conditions which guarantee that the complex structure extends uniquely across a more general singular set can also be obtained, and this will be the subject of a forthcoming paper.

The question of whether the complex structure extends uniquely across an isolated singular point is clearly a local one, at least as far as S is concerned. However, both the results and the techniques of this section will be central in Section 6, where a global modulus of continuity for a uniformizing map is obtained. A more global approach is therefore taken than is necessary for the results of this section alone.

Again, the local nature of the problem means that there is no essential distinction between plain and non-plain surface paper-foldings. For simplicity of exposition, however, the details of the construction and proof (from Section 5.3 onwards) will be carried out in the plain case: the minor modifications needed for non-plain examples will be described at the end of Section 5.6.

The main theorem is stated in Section 5.1, and examples of its application are given in Section 5.2. The idea of the proof (which was inspired by similar constructions in [8, 6, 5]; see also [4]) is to construct a nested sequence of annuli with divergent module sum zooming down to the singular

point q , and to apply Lemmas 22 and 23. In order to be able to construct these annuli in a way which makes it possible to estimate their modules, a *foliated collaring* of the polygon P is used: this is described in Section 5.3. The annuli themselves are constructed in Section 5.4, and some technical lemmas needed to estimate their modules are given in Section 5.5, before the proof of Theorem 59 is given in Section 5.6.

5.1. Statement of results.

Definitions 57 ($m(q; r)$, $n(q; r)$, planar radius). Let G be the scar of a paper-folding scheme (P, \mathcal{P}) , $q \in G$, and $r > 0$. Recall (Notation 10) that $C_G(q; r)$ denotes the set of points of G at d_G -distance exactly r from q . Define

$$n(q; r) := \#C_G(q; r) \in \mathbb{N} \cup \{\infty\}.$$

A radius $r > 0$ is said to be *planar* for q , or *q -planar*, if all points of $C_G(q; r)$ are planar.

Define also

$$m(q; r) := m_G(B_G(q; r)) \in (0, m_G(G)].$$

In the statement and proof of the following result, recall (Definition 51 and Lemma 52) that \bar{r} denotes the injectivity radius of G , and that $\overline{B}_G(q; r)$ is a dendrite for all $q \in G$ and all $r < \bar{r}$.

Lemma 58. *If $r \in (0, \bar{r})$ is a planar radius for $q \in G$, then $n(q; r)$ is finite and is locally constant on both variables q, r . Moreover, given $q \in G$, the set of radii which are not planar for q is a closed subset of $(0, \bar{r})$ of measure zero.*

Proof. Recall (Lemma 33a)) that the set $\overline{\mathcal{V}}$ of non-planar points is a compact subset of G with zero m_G -measure. If $r > 0$ is planar for q there is therefore some $\delta > 0$ such that $|d(s, q) - r| > \delta$ for all $s \in \overline{\mathcal{V}}$.

It follows that for each $x \in C_G(q; r)$, the ball $I_x = B_G(x; \delta)$ is isometric to an interval of length 2δ . Since $\overline{B}_G(q; r)$ is a dendrite and the metric d_G is strictly intrinsic, I_x cannot contain any other point of $C_G(q; r)$ (otherwise the unique shortest path from q to one of the points would be strictly shorter than that to the other). This establishes that $n(q; r)$ is finite, since $m_G(G)$ is finite.

For the local constantness of $n(q; r)$, observe that if $d_G(q, q') < \delta/4$ and $|r - r'| < \delta/4$, then each I_x contains a point of $C_G(q'; r')$, and hence $n(q'; r') \geq n(q; r)$. On the other hand, since $|d(s, q') - r'| > \delta/2$ for all $s \in \overline{\mathcal{V}}$, the same argument works the other way round to show that $n(q; r) \geq n(q'; r')$ as required.

$\overline{\mathcal{V}}$ has zero m_G -measure, and hence zero μ_G^1 -measure by Lemma 33c). Since the function $G \rightarrow \mathbb{R}$ defined by $s \mapsto d_G(q, s)$ is distance non-increasing, it follows from Lemma 20 that the set $\{d_G(q, s) : s \in \overline{\mathcal{V}}\}$ of non-planar radii for q has zero measure.

Since $\overline{\mathcal{V}}$ is compact and $x \mapsto d_G(q, x)$ is a continuous map $G \rightarrow \mathbb{R}$, the set of non-planar radii together with zero is closed in \mathbb{R} , and hence the set of non-planar radii is closed in $(0, \bar{r})$. \square

It is now possible to state one of the main theorems of this paper:

Theorem 59. *Let (P, \mathcal{P}) be a surface paper-folding scheme with associated paper surface S and scar $G \subset S$. If $q \in G$ is an isolated singular point, then the complex structure on $S \setminus \mathcal{V}^s$ extends uniquely across q provided that*

$$(8) \quad \int_0 \frac{dr}{m(q; r) + r \cdot n(q; r)} = \infty.$$

In particular, if all singular points are isolated and the integral condition above holds for every one of them, then S is a compact Riemann surface.

Remark 60. The integral in the statement diverges at planar points in G and at regular vertices. At a regular k -vertex with $k < \infty$, or at a planar point ($k = 2$), $n(q; r) = k$ and $m(q; r) = 2kr$ for all sufficiently small r . Therefore

$$\int_\varepsilon \frac{dr}{m(q; r) + r \cdot n(q; r)}$$

goes like $-\ln \varepsilon$. The proof of Theorem 59 presented in this section also shows that if $q \in G$ is a non-singular point, then the complex structure on $S \setminus (\mathcal{V}^s \cup \{q\})$ extends uniquely across q .

5.2. Example: ∞ -od singularities. A case of special interest is that of ∞ -od singularities: they have already appeared in Example 35 (see also Example 43), and are common in dynamical applications.

Definition 61 (∞ -od singularity). An isolated singularity $q \in G$ is an ∞ -od singularity if there is some $r_0 > 0$ such that $\overline{B}_G(q; r_0)$ is an ∞ -od (Definition 12).

Suppose that $q \in G$ is an ∞ -od singularity. Then if $0 < r \leq r_0$, every point $x \in C_G(q; r)$ is joined to q by a unique arc \overline{xq} all of whose interior points are planar and whose length is r , and these arcs intersect only at q . By Lemma 33d), $m_G(\overline{xq}) = 2r$. There are as many such arcs in $\overline{B}_G(q; r)$ as there are points in $C_G(q; r)$, from which it follows that $m(q; r) \geq 2r \cdot n(q; r)$. This proves the following

Corollary 62. *For the complex structure on \mathcal{R} to extend uniquely across an ∞ -od singularity $q \in G$ it is sufficient that*

$$(9) \quad \int_0 \frac{dr}{m(q; r)} = \infty.$$

□

Example 63. Consider Example 35. Notice that in the criterion of Corollary 62 only the measure $m(q; r) = m_G(B_G(q; r))$ is taken into account and the way in which the identifications about the ∞ -od singularity q are done is not relevant. Thus there is no distinction between the two ways of arranging the folds in the example, since the resulting scars are isometric. What is important is the asymptotics of the decreasing sequence (a_n) of fold lengths.

Write $N(r) := n(q; r) = \max\{n : a_n \geq r\}$ for $r > 0$. Then

$$(10) \quad m(q; r) = 2r \cdot N(r) + 2 \sum_{n > N(r)} a_n.$$

Suppose first that $a_n \asymp 1/n^k$, for some $k > 1$: that is, $C_1/n^k \leq a_n \leq C_2/n^k$ for some positive constants C_1 and C_2 . In this case, $n \leq (C_1/r)^{1/k}$ implies $a_n \geq r$, and $n > (C_2/r)^{1/k}$ implies $a_n < r$, so that $(C_1/r)^{1/k} \leq N(r) \leq (C_2/r)^{1/k}$. Using (10) with the lower bound for $N(r)$ in the first term, the upper bound for $N(r)$ in the second, and lower bounds for a_n gives $m(q; r) \geq C_3 r^{1-1/k}$ for sufficiently small r , so that

$$\int_0 \frac{dr}{m(q; r)} \leq \int_0 \frac{dr}{C_3 r^{1-1/k}} < \infty.$$

This means the criterion cannot be used to guarantee that the complex structure extends uniquely across the point q , and the authors know no way of determining whether or not it does so extend.

If, on the other hand, $a_n \asymp 1/\lambda^n$ for some $\lambda > 1$, then

$$\frac{\ln(C_1/r)}{\ln \lambda} \leq N(r) \leq \frac{\ln(C_2/r)}{\ln \lambda},$$

and it follows from (10) that

$$\begin{aligned} \frac{1}{2} m(q; r) &\leq \frac{r}{\ln \lambda} \ln \frac{C_2}{r} + \frac{C_2}{\lambda^{N(r)}(\lambda - 1)} \\ &\leq \frac{r}{\ln \lambda} \left(\ln C_2 + \ln \frac{1}{r} \right) + \frac{C_2 r}{C_1(\lambda - 1)} \\ &\leq C_3 r \ln \frac{1}{r} \quad \text{for } r \text{ sufficiently small,} \end{aligned}$$

so that

$$2 \int_0 \frac{dr}{m(q; r)} \geq \int_0 \frac{dr}{C_3 r \ln(1/r)} = \infty.$$

This time Corollary 62 applies and thus the quotient space is a complex sphere.

Remark 64. If the Cantor construction of Example 35 is that of the standard middle thirds Cantor set, then listing the edges of the ∞ -od in decreasing order gives

$$a_n = \frac{1}{6 \cdot 3^k} \quad \text{for } 2^k \leq n < 2^{k+1},$$

which implies that

$$\frac{1}{6} \left(\frac{1}{n} \right)^{\frac{\ln 3}{\ln 2}} \leq a_n \leq \frac{1}{2} \left(\frac{1}{n} \right)^{\frac{\ln 3}{\ln 2}}.$$

Thus this is an example in which the hypothesis of Corollary 62 is not satisfied.

5.3. Foliated collaring of P . Let (P, \mathcal{P}) be a surface paper-folding scheme with associated paper surface S and scar G , and let q be an isolated singular point of G . In order to show that the complex structure on $S \setminus \mathcal{V}^s$ extends uniquely across q under appropriate conditions, a nested sequence of annuli zooming down to q will be defined. These annuli will be constructed in a foliated neighborhood of q arising from a foliated collar of P , which is described in this section.

For simplicity of exposition, the case in which (P, \mathcal{P}) is plain will be considered first. Thus P is a single polygon, G is a dendrite, and S is a topological sphere (Definitions 34 and Theorem 42).

The collar \tilde{Q} of P is constructed as a union of trapezoids whose bases are the sides of P ; whose vertical sides bisect the angles at the vertices of P ; and which have fixed height \bar{h} , chosen small enough that the trapezoids are far from degenerate and intersect only along their vertical sides. It has a *horizontal foliation* by leaves parallel to ∂P , and a *vertical foliation* by leaves joining the base and the top of each trapezoid: see Figure 4. The following paragraphs define the collar and foliations more carefully, and set up the notation which will be used.

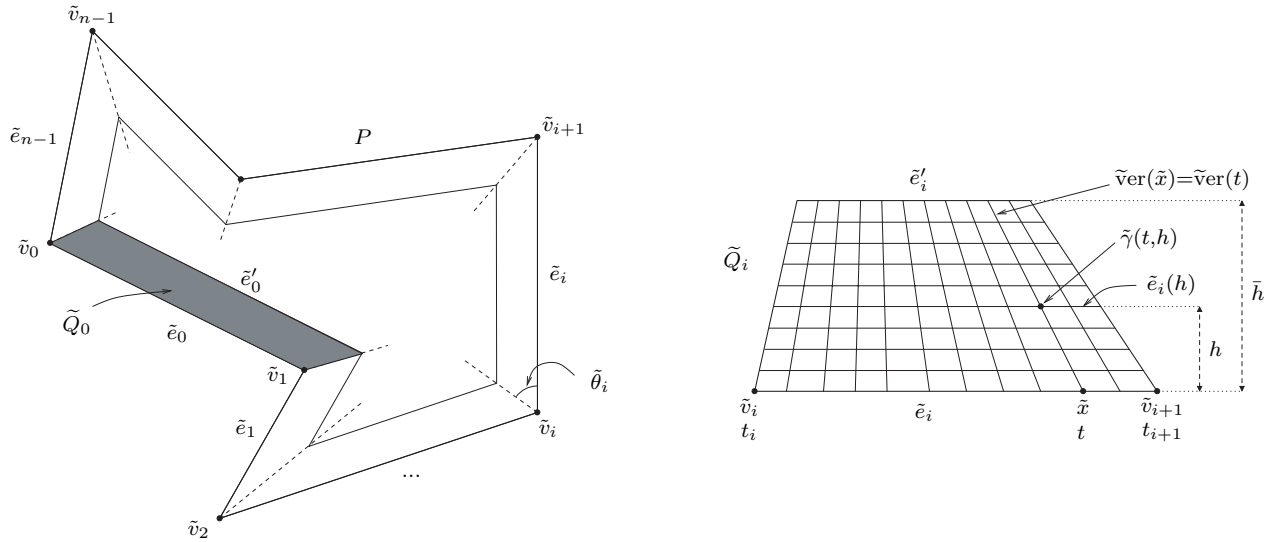


FIGURE 4. The collar \tilde{Q} and its foliations

5.3.1. The collar \tilde{Q} . Choose a labeling \tilde{v}_i ($i = 0, \dots, n-1$) of the vertices of P listed counter-clockwise around ∂P , and let \tilde{e}_i be the side of P with endpoints \tilde{v}_i and \tilde{v}_{i+1} (here and throughout, subscripts on cyclically ordered objects are taken mod n). Write $L = |\partial P|$, and let $\tilde{\gamma}_0: [0, L] \rightarrow \partial P$ be the order-preserving parameterization of ∂P by arc-length with $\tilde{\gamma}_0(0) = \tilde{v}_0$. Denote by $t_i \in [0, L)$ the parameter with $\tilde{\gamma}_0(t_i) = \tilde{v}_i$.

A *trapezoid* is a quadrilateral in \mathbb{R}^2 with two parallel sides, which are called its *base* and its *top*; the other sides are called *vertical sides*. The *height* of the trapezoid is the distance between the parallel lines containing its base and its top.

Pick a height \bar{h} small enough that the trapezoids \tilde{Q}_i which have bases \tilde{e}_i , heights \bar{h} , and vertical sides along the rays bisecting the internal angles of P satisfy:

- a) The lengths of the tops of the trapezoids are between half and twice the lengths of their bases;
and
- b) The trapezoids intersect only along their vertical sides.

This height \bar{h} is an important quantity in the construction, and will remain fixed throughout the remainder of the section. Denote the top of \tilde{Q}_i by \tilde{e}'_i , and let $\tilde{\theta}_i$ be half of the internal angle of ∂P at \tilde{v}_i : thus the internal angles of \tilde{Q}_i at the endpoints of its base are $\tilde{\theta}_i$ and $\tilde{\theta}_{i+1}$. Condition a) above is that

$$(11) \quad \frac{1}{2} \leq \frac{|\tilde{e}'_i|}{|\tilde{e}_i|} \leq 2 \quad \text{for } i = 0, \dots, n-1.$$

Let

$$\tilde{Q} = \bigcup_{i=0}^{n-1} \tilde{Q}_i,$$

a closed collar neighborhood of ∂P in P .

5.3.2. The foliations $\tilde{\text{Hor}}$ and $\tilde{\text{Ver}}$. For each $h \in [0, \bar{h}]$, let $\tilde{e}_i(h) \subset \tilde{Q}_i$ be the segment parallel to the base of \tilde{Q}_i at height h , so that $\tilde{e}_i(0) = \tilde{e}_i$ and $\tilde{e}_i(\bar{h}) = \tilde{e}'_i$. Then the union $\tilde{\text{hor}}(h)$ of these segments is a polygonal simple closed curve: these simple closed curves are the leaves of the *horizontal foliation*

$$\tilde{\text{Hor}} = \left\{ \tilde{\text{hor}}(h) : h \in [0, \bar{h}] \right\}$$

of \tilde{Q} . The parameter h is called the *height* of the leaf $\tilde{\text{hor}}(h)$.

Write

$$\tilde{Q}(h) = \bigcup_{h' \in [0, h]} \tilde{\text{hor}}(h'),$$

the subset of \tilde{Q} consisting of leaves with heights not exceeding h : $\tilde{Q}(h) \subset \tilde{Q} = \tilde{Q}(\bar{h})$ is therefore also a closed collar neighborhood of ∂P for each $h \in (0, \bar{h}]$.

To construct the vertical foliation, let $\varphi_i: \tilde{e}_i \rightarrow \tilde{e}'_i$ be the orientation-preserving scaling from \tilde{e}_i to \tilde{e}'_i . For each $\tilde{x} = \tilde{\gamma}_0(t) \in \tilde{e}_i$, denote by $\tilde{\text{ver}}(\tilde{x})$ or $\tilde{\text{ver}}(t)$ the straight line segment which joins \tilde{x} to $\varphi_i(\tilde{x})$. These segments are the leaves of the *vertical foliation*

$$\tilde{\text{Ver}} = \{ \tilde{\text{ver}}(t) : t \in [0, L] \}$$

of \tilde{Q} .

Define $\tilde{\theta}: [0, L] \setminus \{t_0, \dots, t_{n-1}\} \rightarrow (0, \pi)$ by setting $\tilde{\theta}(t)$ to be the angle between ∂P and $\tilde{\text{ver}}(t)$ at $\tilde{\gamma}_0(t)$: that is, the angle between the oriented side of ∂P containing $\tilde{\gamma}_0(t)$ and the leaf $\tilde{\text{ver}}(t)$ pointing into P . This function has well-defined limits as t approaches each t_i from the left or the right: $\tilde{\theta}(t_i^-) = \pi - \tilde{\theta}_i$, and $\tilde{\theta}(t_i^+) = \tilde{\theta}_i$. The notation $\tilde{\theta}(\tilde{x}) = \tilde{\theta}(t)$ will also be used when $\tilde{x} = \tilde{\gamma}_0(t)$.

The foliations $\tilde{\text{Hor}}$ and $\tilde{\text{Ver}}$ yield a parameterization

$$\tilde{\gamma}: [0, L) \times [0, \bar{h}] \rightarrow \tilde{Q}$$

of \tilde{Q} , where $\tilde{\gamma}(t, h)$ is the unique point of $\tilde{\text{ver}}(t) \cap \tilde{\text{hor}}(h)$.

For each $h \in (0, \bar{h}]$, denote by $\tilde{\psi}_h: \tilde{Q}(h) \rightarrow \partial P$ the retraction of $\tilde{Q}(h)$ onto ∂P which slides each point along its vertical leaf:

$$\tilde{\psi}_h(\gamma(t, h')) = \gamma(t, 0) \quad (\text{all } t \in [0, L) \text{ and } h' \in [0, h]).$$

In particular, $\tilde{\psi}_{\bar{h}}$ is a retraction which squashes all of \tilde{Q} onto ∂P .

5.3.3. The foliations on S . The projections to the paper surface S of the structures defined above are denoted by removing tildes (see Figure 5). Thus $Q = \pi(\tilde{Q})$ is a closed disk neighborhood of the scar G , and similarly $Q(h) = \pi(\tilde{Q}(h))$ is a closed subdisk neighborhood for each $h \in (0, \bar{h}]$. Q has horizontal and vertical foliations $\text{Hor} = \pi(\tilde{\text{Hor}})$ and $\text{Ver} = \pi(\tilde{\text{Ver}})$. The leaves of Hor are projections of leaves of $\tilde{\text{Hor}}$: $\text{hor}(h) = \pi(\tilde{\text{hor}}(h))$. The leaves of Ver , however, are unions of projections of leaves of $\tilde{\text{Ver}}$: for each $x \in G$, the leaf of Ver containing x is defined to be

$$\text{ver}(x) := \bigcup \{ \pi(\tilde{\text{ver}}(\tilde{x})) : \tilde{x} \in \pi^{-1}(x) \}.$$

Thus $\text{ver}(x)$ is an arc if and only if $\#\pi^{-1}(x) \leq 2$. If x is a k -vertex for $k > 2$ then $\text{ver}(x)$ is a star with k branches. Note, however, that if x is an ∞ -vertex then $\text{ver}(x)$ is not an ∞ -od in the sense of Definition 12, since the lengths of its branches do not converge to zero.

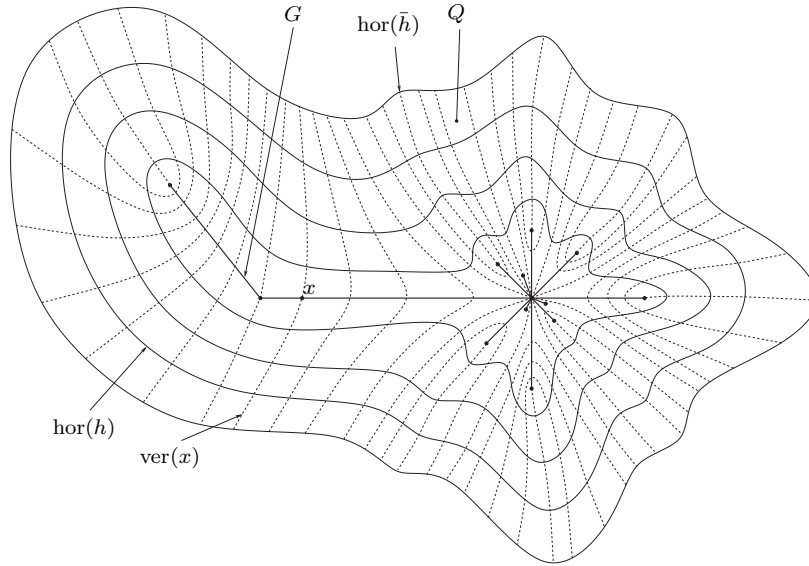


FIGURE 5. The foliations projected to S

The disks $Q(h)$ for $0 < h \leq \bar{h}$ are similarly foliated by horizontal leaves $\text{hor}(h')$ with $0 < h' \leq h$, and vertical leaves $\text{ver}_h(x)$, which are the leaves $\text{ver}(x)$ trimmed at their intersection with $\text{hor}(h)$.

The composition $\gamma := \pi \circ \tilde{\gamma}: [0, L) \times [0, \bar{h}] \rightarrow Q$ parameterizes Q , although it is not injective on G . Notice that because the retractions $\tilde{\psi}_h: \tilde{Q}(h) \rightarrow \partial P$ fix ∂P pointwise, the compositions

$$\psi_h := \pi \circ \tilde{\psi}_h \circ \pi^{-1}: Q(h) \rightarrow G$$

are well-defined retractions of $Q(h)$ onto G .

5.4. The system of annuli $\text{Ann}(q; r, s)$. Let q be a point of the scar G . In this section annuli $\text{Ann}(q; r, s)$ about q will be constructed for each pair of q -planar radii $r < s$. The annuli will be defined as differences of two topological closed disks: $\text{Ann}(q; r, s) = \text{Int}(D(q; s)) \setminus D(q; r)$.

There are naturally two parameters r and h involved in constructing such disks about q using the foliations of Q , which describe respectively the vertical and the horizontal leaves which will form its boundary. Here, though, the ratio between these parameters will be fixed. The ratio is determined by a number $\bar{r} > 0$, which in this section can be chosen arbitrarily: in the non-plain case, however, it will become the injectivity radius of G (Definition 51).

Choose $\bar{r} > 0$, then, and define a function $h: [0, \bar{r}] \rightarrow [0, \bar{h}/2]$ by

$$h(r) := \left(\frac{\bar{h}}{2\bar{r}} \right) r,$$

which will fix the parameter h given the radius r .

Definition 65 ($D(q; r)$). Let $q \in G$ and let $r \in (0, \bar{r}]$. The subset $D(q; r)$ of Q is defined by

$$D(q; r) := \psi_{h(r)}^{-1}(\overline{B}_G(q; r)).$$

Alternatively, $D(q; r)$ is the intersection of $Q(h(r))$ with the union of the vertical leaves $\text{ver}(x)$ with $d_G(q, x) \leq r$.

Lemma 66. *Let $q \in G$ and $r \in (0, \bar{r}]$ be a q -planar radius. Write $n = n(q; r)$ and $h = h(r)$. Then $D(q; r)$ is a topological closed disk, whose boundary is composed of n disjoint subarcs of the horizontal leaf $\text{hor}(h)$, and the n trimmed vertical leaves $\text{ver}_h(x)$ with $x \in C_G(q; r)$.*

Proof. (See Figure 6.) Write $C_G(q; r) = \{x_0, \dots, x_{n-1}\}$. By definition, the boundary $\partial_{Q(h)} D(q; r)$ of $D(q; r) = \psi_h^{-1}(\overline{B}_G(q; r)) \subset Q(h)$ in $Q(h)$ is contained in $\psi_h^{-1}(C_G(q; r))$. Moreover, since r is q -planar, every neighborhood of each point of $C_G(q; r)$ contains both points which are closer to q and points which are further away (cf. the proof of Lemma 58), and hence

$$\partial_{Q(h)} D(q; r) = \psi_h^{-1}(C_G(q; r)) = \bigcup_{i=0}^{n-1} \psi_h^{-1}(x_i).$$

Since the points x_i are planar, each $\psi_h^{-1}(x_i) = \text{ver}_h(x_i)$ is an arc which intersects $\text{hor}(h) = \partial_S Q(h)$ exactly at its endpoints: that is, a cross cut in $Q(h)$. For each i , $Q(h) \setminus \text{ver}_h(x_i)$ has

exactly two components, one of which intersects G in the complement of $\overline{B}_G(q; r)$. Therefore every $\text{ver}_h(x_j)$ with $j \neq i$ is contained in the same component as q . It follows that $\partial_S D(q; r)$ is the simple closed curve composed of the arcs $\text{ver}_h(x_i)$ and the n subarcs of $\text{hor}(h)$ joining the endpoints of consecutive cross cuts in the cyclic order around $\text{hor}(h)$. \square

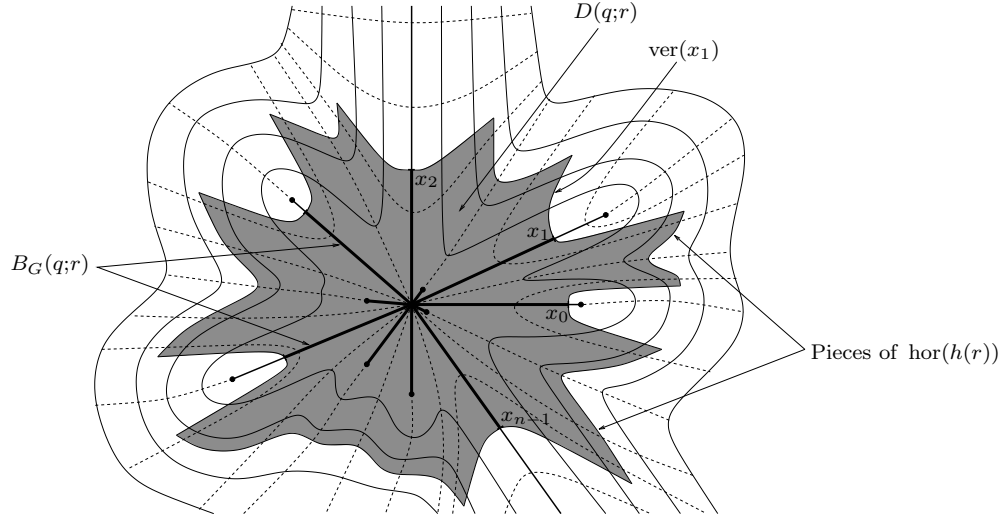


FIGURE 6. The disk $D(q; r)$

The annuli which will be used in the proof of Theorem 59 can now be defined.

Definition 67 ($\text{Ann}(q; r, s)$). Let $q \in G$, and $r, s \in (0, \bar{r}]$ be q -planar radii with $r < s$. The subset $\text{Ann}(q; r, s)$ of Q is defined by

$$\text{Ann}(q; r, s) = \text{Int}(D(q; s)) \setminus D(q; r).$$

By Lemma 66, and since $D(q; r) \subset \text{Int}(D(q; s))$, $\text{Ann}(q; r, s)$ is an open annular region with q in its bounded complementary component (the complementary component not containing ∂Q).

5.5. Polygon constants and geometry of the trapezoid construction. The goal now is to find lower bounds on the conformal modules of the annuli $\text{Ann}(q; r, s)$. In order to use (3), it is necessary to estimate the distance in S between the two boundary components of the annulus, and the area of the annulus. This will be done by lifting to the polygon P , where the preimage of the disk $D(q; r)$ is a union of *polytrapezoids*.

The bounds will be expressed in terms of the *polygon constants* of P : this will make it possible, in Section 6, to obtain uniform moduli of continuity for uniformizing maps in families of polygons with bounded constants.

Definition 68 (Polygon constants). The *polygon constants* of P are the numbers \bar{h} , \bar{r} , and $|\partial P|$.

Remarks 69.

- a) Of course there is some freedom in the choice of \bar{h} , and total freedom in the choice of \bar{r} in the plain case. The important point is that, in order to prevent the constructions from becoming degenerate in families of examples, \bar{r} and \bar{h} should be bounded away from zero, and $|\partial P|$ should be bounded above.
- b) The internal semi-angles $\tilde{\theta}_i$ of P are bounded away from 0 and π in terms of the polygon constants, since the vertical edge of \tilde{Q}_{i-1} and \tilde{Q}_i emanating from \tilde{v}_i has length $\bar{h}/\sin \tilde{\theta}_i$ and is contained in P : hence

$$(12) \quad \sin \tilde{\theta}_i \geq \frac{2\bar{h}}{|\partial P|} \quad \text{for } i = 0, \dots, n-1.$$

The following elementary plane geometry lemma bounds the derivative of the cotangent of the function $\tilde{\theta}(\tilde{x})$ of Section 5.3.2.

Lemma 70. *Let \tilde{x} lie in the interior of one of the sides \tilde{e}_i of P . Then*

$$\left| \frac{d}{d\tilde{x}} \cot \tilde{\theta}(\tilde{x}) \right| \leq \frac{1}{\bar{h}}.$$

Proof. Suppose first that $|\tilde{e}_i| > |\tilde{e}'_i|$. Extend the vertical sides of the trapezoid \tilde{Q}_i to form a triangle of height H : by (11), $H \geq 2\bar{h}$. The vertical leaf through $\tilde{x} \in \tilde{e}_i$ is a segment of the straight line joining \tilde{x} to the apex of this triangle.

Now if \tilde{x}_1, \tilde{x}_2 lie in \tilde{e}_i , then the distance between \tilde{x}_1 and \tilde{x}_2 is given by $H|\cot(\tilde{\theta}(\tilde{x}_1)) - \cot(\tilde{\theta}(\tilde{x}_2))|$. Hence

$$\left| \frac{d}{d\tilde{x}} \cot \tilde{\theta}(\tilde{x}) \right| = \frac{1}{H} \leq \frac{1}{2\bar{h}}.$$

If $|\tilde{e}_i| < |\tilde{e}'_i|$, then again extend the vertical sides of \tilde{Q}_i to form a triangle of height $H \geq 2\bar{h}$. In this case the distance between \tilde{x}_1 and \tilde{x}_2 is given by $(H - \bar{h})|\cot(\tilde{\theta}(\tilde{x}_1)) - \cot(\tilde{\theta}(\tilde{x}_2))|$, so that

$$\left| \frac{d}{d\tilde{x}} \cot \tilde{\theta}(\tilde{x}) \right| = \frac{1}{H - \bar{h}} \leq \frac{1}{\bar{h}}.$$

If $|\tilde{e}_i| = |\tilde{e}'_i|$, then $\tilde{\theta}(\tilde{x})$ is constant on \tilde{e}_i . □

Now let q be a point of G , which will remain fixed throughout the remainder of this section (so the dependence of many objects on q will not be explicitly noted). Let $(r_1, r_2) \subset (0, \bar{r}]$ be an interval of q -planar radii, and choose $r \in (r_1, r_2)$. As before (see Lemma 66 and its proof), write $n = n(q; r)$, and let $C_G(q; r) = \{x_0(r), \dots, x_{n-1}(r)\}$, labelling its points in the counterclockwise direction around $\text{hor}(h(r))$.

Since G is a dendrite, each of the points $x_i(r)$ is connected to $x_{i+1}(r)$ by a unique arc $\alpha_i = [x_i(r), x_{i+1}(r)]_G$ in G (Theorem 8b)). These arcs, which are cross cuts in $D(q; r)$, form the boundaries in $D(q; r)$ of n closed subdisks $T_i(q; r)$ ($0 \leq i \leq n-1$) (see Figure 7). As a subset of S , $T_i(q; r)$ is bounded by:

- The arc α_i ;
- $\text{ver}_{h(r)}^+(x_i(r))$ and $\text{ver}_{h(r)}^-(x_{i+1}(r))$, segments of vertical leaf with one endpoint at $x_i(r)$ and $x_{i+1}(r)$ respectively – denote their other endpoints $z_i^+(r)$ and $z_{i+1}^-(r)$; and
- $\Lambda_i(r)$, an arc of $\text{hor}(h(r))$ from $z_i^+(r)$ to $z_{i+1}^-(r)$ in the counterclockwise direction.

Since the union of the arcs α_i is a tree whose set of endpoints is $C_G(q; r) \subset \partial D(q; r)$,

$$(13) \quad D(q; r) = \bigcup_{i=0}^{n-1} T_i(q; r),$$

and the subsets $T_i(q; r)$ intersect only along the arcs α_i . (If a topological disk D contains a tree T which intersects ∂D precisely at its endpoints, then components of $D \setminus T$ correspond bijectively to pairs of consecutive endpoints on ∂D .)

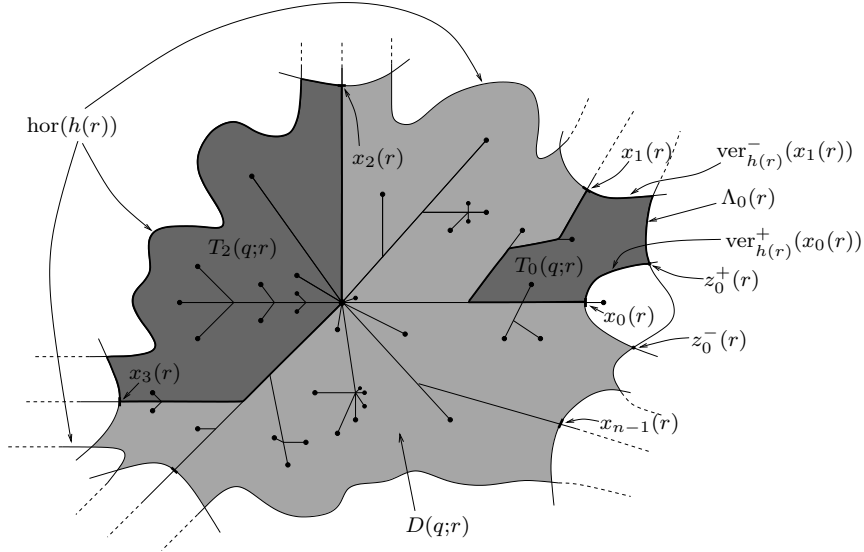


FIGURE 7. The regions $T_i(q; r)$

Now $\Lambda_i(r)$ lies in $Q \setminus G$, and hence has a well-defined preimage $\tilde{\Lambda}_i(r) = \pi^{-1}(\Lambda_i(r))$, which is an arc of $\tilde{\text{hor}}(h(r))$ whose initial and final endpoints in the *clockwise* direction along $\tilde{\text{hor}}(h(r))$ are $\tilde{z}_i^+(r)$ and $\tilde{z}_{i+1}^-(r)$, the preimages of $z_i^+(r)$ and $z_{i+1}^-(r)$ (see Figure 8). Similarly, $\text{ver}_{h(r)}^+(x_i(r))$ and $\text{ver}_{h(r)}^-(x_{i+1}(r))$ have well-defined preimage arcs $\tilde{\text{ver}}_{h(r)}(\tilde{x}_i^+(r))$ and $\tilde{\text{ver}}_{h(r)}(\tilde{x}_{i+1}^-(r))$, with endpoints $\tilde{x}_i^+(r)$ and $\tilde{x}_{i+1}^-(r)$ in ∂P .

Write $\tilde{\Delta}_i(r)$ for the union $\tilde{\Lambda}_i(r) \cup \tilde{\text{ver}}_{h(r)}(\tilde{x}_i^+(r)) \cup \tilde{\text{ver}}_{h(r)}(\tilde{x}_{i+1}^-(r))$, so that

$$\pi \left(\bigcup_{i=0}^{n-1} \tilde{\Delta}_i(r) \right) = \partial D(q; r).$$

Suppose that $\tilde{x}_{i+1}^-(r) \in \tilde{e}_k$ and $\tilde{x}_i^+(r) \in \tilde{e}_\ell$. The arc $\tilde{L}_i(r)$ of ∂P from $\tilde{x}_{i+1}^-(r)$ to $\tilde{x}_i^+(r)$ in the counterclockwise direction therefore contains the sides $\tilde{e}_{k+1}, \dots, \tilde{e}_{\ell-1}$, and segments of the sides \tilde{e}_k and \tilde{e}_ℓ . The *polytrapezoid* (union of trapezoids of the same height glued along their vertical sides) $\tilde{T}_i(q; r)$ bounded by $\tilde{L}_i(r)$ and $\tilde{\Delta}_i(r)$ satisfies $\pi(\tilde{T}_i(q; r)) = T_i(q; r)$, and $\pi: \tilde{T}_i(q; r) \rightarrow T_i(q; r)$ is a homeomorphism away from $\tilde{L}_i(r)$. Notice that

$$\bigcup_{i=0}^{n-1} \tilde{L}_i(r) = \pi^{-1}(\overline{B}_G(q; r)),$$

and hence

$$(14) \quad \sum_{i=0}^{n-1} |\tilde{L}_i(r)| = m_G(\overline{B}_G(q; r)) = m(q; r).$$

Denote by $\theta_0(r)$ and $\theta_1(r)$ the internal angles of the polytrapezoid at the vertices $\tilde{x}_{i+1}^-(r)$ and $\tilde{x}_i^+(r)$ respectively (the other internal angles along $\tilde{L}_i(r)$ are $2\tilde{\theta}_{k+1}, \dots, 2\tilde{\theta}_\ell$, independent of $r \in (r_1, r_2)$).

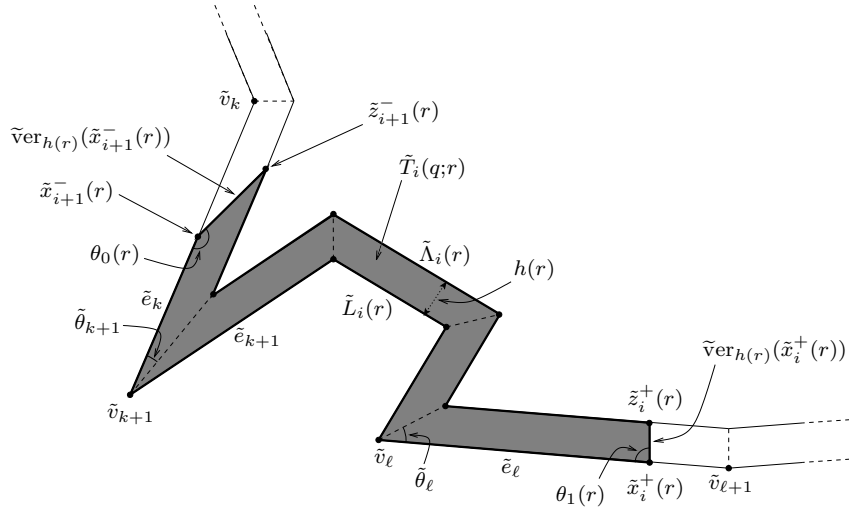


FIGURE 8. The polytrapezoid $\tilde{T}_i(q; r)$

Lemma 71.

$$\text{Area}(\tilde{T}_i(q; r)) = h(r) |\tilde{L}_i(r)| - \frac{h(r)^2}{2} \left(\cot \theta_0(r) + \cot \theta_1(r) + 2 \sum_{j=k+1}^{\ell} \cot \tilde{\theta}_j \right).$$

Proof. Observe that if a trapezoid has base e , top e' , height h , and internal angles θ_1 and θ_2 at its base vertices, then

$$(15) \quad |e'| = |e| - h(\cot \theta_1 + \cot \theta_2).$$

The result follows immediately on summing the areas $h(|e| + |e'|)/2$ of the constituent trapezoids of $\tilde{T}_i(q; r)$. \square

Lemma 72.

$$0 < \frac{d}{dr} \text{Area} \left(\tilde{T}_i(q; r) \right) \leq \frac{5\bar{h}}{4\bar{r}} \left(\left| \tilde{L}_i(r) \right| + r \right).$$

Proof. The first inequality in the statement is obvious.

Differentiating the expression for the area given by Lemma 71, recalling that $h(r) = (\bar{h}/2\bar{r})r$, and observing that $\frac{d}{dr} \left| \tilde{L}_i(r) \right| = 2$ and that

$$\left| \tilde{\Lambda}_i(r) \right| = \left| \tilde{L}_i(r) \right| - h(r) \left(\cot \theta_0(r) + \cot \theta_1(r) + 2 \sum_{j=k+1}^{\ell} \cot \tilde{\theta}_j \right)$$

by (15), gives

$$\begin{aligned} \frac{d}{dr} \text{Area} \left(\tilde{T}_i(q; r) \right) &= \frac{dh(r)}{dr} \left[\left| \tilde{L}_i(r) \right| - h(r) \left(\cot \theta_0(r) + \cot \theta_1(r) + 2 \sum_{j=k+1}^{\ell} \cot \tilde{\theta}_j \right) \right] \\ &\quad + h(r) \left[\frac{d}{dr} \left| \tilde{L}_i(r) \right| - \frac{h(r)}{2} \frac{d}{dr} (\cot \theta_0(r) + \cot \theta_1(r)) \right] \\ &\leq \frac{\bar{h}}{2\bar{r}} \left| \tilde{\Lambda}_i(r) \right| + \frac{\bar{h}r}{2\bar{r}} \left(2 + \frac{h(r)}{\bar{h}} \right) \\ &\leq \frac{\bar{h}}{\bar{r}} \left| \tilde{L}_i(r) \right| + \frac{5\bar{h}}{4\bar{r}} r \end{aligned}$$

as required, using $\left| \tilde{\Lambda}_i(r) \right| \leq 2 \left| \tilde{L}_i(r) \right|$ (which follows from (11)) and Lemma 70. \square

In order to estimate the d_S -distance between the boundary components of $\text{Ann}(q; r, s)$, where $r_1 < r < s < r_2$, it is necessary to find a lower bound for the distance between $\tilde{\Delta}_i(r)$ and $\tilde{\Delta}_i(s)$, which project to part of the inner and the outer boundary components of $\text{Ann}(q; r, s)$ respectively.

Lemma 73. *Let (r_1, r_2) be an interval of q -planar radii. If $r_1 < r < s < r_2$ then*

$$d \left(\tilde{\Delta}_i(r), \tilde{\Delta}_i(s) \right) \geq (s - r) \min \left\{ \frac{\bar{h}}{2\bar{r}}, \frac{1}{2} \right\}.$$

Proof. Let $\tilde{p} \in \tilde{\Delta}_i(s)$. If $\tilde{p} \in \tilde{\Delta}_i(r)$, then every point of $\tilde{\Delta}_i(r)$ is distance at least $h(s) - h(r) = \frac{\bar{h}}{2\bar{r}}(s - r)$ from \tilde{p} .

If $\tilde{p} \in \tilde{\text{ver}}_{h(s)}(\tilde{x}_i^+(s))$, then no point of $\tilde{\Delta}_i(r)$ can lie closer to \tilde{p} than the closest point of $\tilde{\text{ver}}_{h(s)}(\tilde{x}_i^+(r))$ (the arc obtained by extending $\tilde{\text{ver}}_{h(r)}(\tilde{x}_i^+(r)) \subset \tilde{\Delta}_i(r)$ to $\tilde{\text{hor}}(h(s))$). Hence $d(\tilde{p}, \tilde{\Delta}_i(r)) \geq (s - r)/2$ by (11). The proof if $\tilde{p} \in \tilde{\text{ver}}_{h(s)}(\tilde{x}_{i+1}^-(s))$ is similar. \square

5.6. Proof of Theorem 59. Initially in this section, (P, \mathcal{P}) is again a plain paper-folding scheme with associated paper sphere S and scar $G \subset S$, and $q \in G$. The first step is to interpret the geometric results of the previous section in the sphere S .

Lemma 74. *For all planar radii $r \in (0, \bar{r})$,*

$$0 < \frac{d}{dr} \text{Area}_S(D(q; r)) \leq \frac{5\bar{h}}{4\bar{r}} (m(q; r) + r \cdot n(q; r)).$$

Proof. Write $n = n(q; r)$. From (13) it follows that

$$\text{Area}_S(D(q; r)) = \sum_{i=0}^{n-1} \text{Area}_{\mathbb{R}^2}(\tilde{T}_i(q; r)).$$

Hence, by Lemma 72,

$$\begin{aligned} 0 < \frac{d}{dr} \text{Area}_S(D(q; r)) &\leq \frac{5\bar{h}}{4\bar{r}} \sum_{i=0}^{n-1} \left(|\tilde{L}_i(r)| + r \right) \\ &= \frac{5\bar{h}}{4\bar{r}} (m(q; r) + r \cdot n(q; r)), \end{aligned}$$

by (14) as required. \square

Lemma 75. *Let (r_1, r_2) be an interval of q -planar radii, and $r_1 < r < s < r_2$. Let $\xi(r)$ and $\xi(s)$ denote the boundary components of $\text{Ann}(q; r, s)$. Then*

$$d_S(\xi(r), \xi(s)) \geq (s - r) \min \left\{ \frac{\bar{h}}{2\bar{r}}, \frac{1}{2} \right\}.$$

Proof. Let $D := d_S(\xi(r), \xi(s))$, and let $p' \in \xi(r)$ and $q' \in \xi(s)$ be points with $d_S(p', q') = D$. Since d_S is strictly intrinsic by Theorem 56b), there is an arc γ' from p' to q' of length D , and apart from its endpoints this arc, being the shortest from $\xi(r)$ to $\xi(s)$, must lie in $\text{Ann}(q; r, s)$.

It will be shown that, for any $r \leq u < v \leq s$, the length of any arc γ from $\xi(u)$ to $\xi(v)$ with $\gamma \subset \text{Ann}(q; u, v)$ is at least $C(v - u)$, where $C = \min \left\{ \frac{\bar{h}}{2\bar{r}}, \frac{1}{2} \right\}$, which will establish the result.

It can be assumed without loss of generality that there are no subarcs of γ whose endpoints lie on the same component of $G \cap \overline{\text{Ann}(q; u, v)}$ and whose interiors lie entirely in the interior of a single $T_i(q; v)$, since such a subarc could be replaced with a shorter subarc lying in G . Therefore

the intersection of $\overset{\circ}{\gamma}$ with G has a finite number N of components. The result will be shown by induction on N .

If $N = 0$, then the arc lies in a single $T_i(q; v)$, and the result is immediate by Lemma 73. If $N > 0$, then let $t_1 = \inf\{t > 0 : \gamma(t) \in G\}$ and $t_2 = \inf\{t > t_1 : \gamma(t) \notin G\}$, so that $\gamma([t_1, t_2]) \subset G$. Split γ into three subarcs $\gamma_1: [0, t_1] \rightarrow S$, $\gamma_2: [t_1, t_2] \rightarrow S$, and $\gamma_3: [t_2, 1] \rightarrow S$ (one of these arcs may degenerate to a point). Let $d(q, \gamma(t_1)) = w_1 \in [u, v]$ and $d(q, \gamma(t_2)) = w_2 \in [u, v]$. By the inductive hypothesis, γ_1 has length at least $C(w_1 - u)$ and γ_3 has length at least $C(v - w_2)$, while γ_2 lies entirely in a single $T_i(q; w_2)$ and hence has length at least $C(w_2 - w_1)$ by Lemma 73. \square

Lemma 76. *Let r and s be q -planar radii with $0 < r < s < \bar{r}$. Then*

$$\text{mod Ann}(q; r, s) \geq \int_r^s \frac{C^2 \, du}{\text{Area}'_S(D(q; u))},$$

where

$$C := \min \left\{ \frac{\bar{h}}{2\bar{r}}, \frac{1}{2} \right\} \quad \text{and} \quad \text{Area}'_S(D(q; r)) := \frac{d \, \text{Area}_S(D(q; r))}{dr}.$$

Proof. Let $[r_1, r_2]$ be an interval of q -planar radii. Then

$$\text{mod Ann}(q; r_1, r_2) \geq \frac{C^2(r_2 - r_1)^2}{\text{Area}_S(\text{Ann}(q; r_1, r_2))}$$

by (3) and Lemma 75.

For any partition $r_1 = s_0 < s_1 < \dots < s_k = r_2$ of $[r_1, r_2]$, it follows from (2) that

$$\begin{aligned} \text{mod Ann}(q; r_1, r_2) &\geq \sum_{j=1}^k \text{mod Ann}(q; s_{j-1}, s_j) \\ &\geq C^2 \sum_{j=1}^k \frac{(s_j - s_{j-1})^2}{\text{Area}_S(D(q; s_j)) - \text{Area}_S(D(q; s_{j-1}))}, \end{aligned}$$

and taking the supremum over all partitions of $[r_1, r_2]$ gives

$$\text{mod Ann}(q; r_1, r_2) \geq \int_{r_1}^{r_2} \frac{C^2 \, du}{\text{Area}'_S(D(q; u))},$$

since $\text{Area}'_S(D(q; u))$ is continuous and positive at planar radii. Hence for any collection of disjoint intervals $[r_1^k, r_2^k] \subset [r, s]$ of q -planar radii, (2) gives

$$\text{mod Ann}(q; r, s) \geq \sum_k \int_{r_1^k}^{r_2^k} \frac{C^2 \, du}{\text{Area}'_S(D(q; u))}.$$

Taking the supremum over such collections of disjoint intervals, using the fact that the set of non-planar radii is closed and has zero measure, gives the result. \square

Combining the results of Lemmas 76 and 74 gives that, for all q -planar radii $0 < r < s < \bar{r}$,

$$(16) \quad \text{mod Ann}(q; r, s) \geq \int_r^s \frac{C^2 \, du}{\frac{5\bar{h}}{4\bar{r}} (m(q; r) + r \cdot n(q; r))},$$

where $C = \min \left\{ \frac{\bar{h}}{2\bar{r}}, \frac{1}{2} \right\}$. This motivates the following definition:

Definition 77 (Paper-folding goodness function). Let (P, \mathcal{P}) be a surface paper folding scheme, and let P have polygon constants \bar{h} and \bar{r} (and $|\partial P|$). Let

$$M = \frac{1}{5} \min \left\{ \frac{\bar{r}}{\bar{h}}, \frac{\bar{h}}{\bar{r}} \right\},$$

and define a function $\iota: G \times (0, \bar{r}) \rightarrow [0, \infty)$ by

$$\iota(q; r) := \begin{cases} \frac{M}{m(q; r) + r \cdot n(q; r)} & \text{if } n(q; r) < \infty, \\ 0 & \text{otherwise.} \end{cases}$$

ι is called a *paper-folding goodness function* for (P, \mathcal{P}) .

Remarks 78. If r is a q -planar radius then ι is non-zero and continuous at $(q; r)$ by Lemma 58 (it is clear that $m(q; r)$ is continuous at such a $(q; r)$). Hence, using Lemma 58 again, for each $q \in G$ the set of radii r at which $\iota(q; r) = 0$ or $\iota(q; r)$ is discontinuous has measure zero.

As mentioned before, goodness functions are not uniquely determined since there is some freedom in the choice of polygon constants. What is important about them in this section is that their integrals down to zero be divergent. The rate of divergence will be important in Section 6, where moduli of continuity are discussed.

Inequality (16) now becomes:

$$(17) \quad \text{mod Ann}(q; r, s) \geq \int_r^s \iota(q; t) \, dt$$

for every pair of planar radii $0 < r < s < \bar{r}$.

Proof of Theorem 59. Let $q \in G$ be an isolated singular point. Pick a q -planar radius $r_0 \in (0, \bar{r})$ small enough that q is the only singular point in $D(q; r_0)$, and let $W = D(q; r_0) \setminus \{q\}$. Then W is a Riemann surface, since it is contained in $S \setminus \mathcal{V}^s$ where there is a well-defined conformal structure. Since W is topologically a disk minus a point, it is conformally homeomorphic to a plane domain (see [2], for example). In order to prove that q is a puncture of the complex structure on W , observe that (17) yields, for q -planar radii $0 < r < r_0$,

$$\text{mod Ann}(q; r, r_0) \geq \int_r^{r_0} \iota(q; s) \, ds.$$

Since this integral diverges as $r \searrow 0$ by hypothesis, the result follows from Lemma 22 and Riemann's Removable Singularity Theorem.

Only minor modifications are needed to treat the case of non-plain surface paper-folding schemes (P, \mathcal{P}) . The constant \bar{h} should be chosen in such a way that foliated collars of height \bar{h} can be constructed for all of the polygons P , and the constant \bar{r} should be chosen to be the injectivity radius of G . It then follows from Lemma 52 that $\overline{B}_G(q; r)$ is a dendrite for all $q \in G$ and all $r < \bar{r}$, and the constructions and proof go through exactly as in the plain case. \square

6. MODULUS OF CONTINUITY

Let (P, \mathcal{P}) be a surface paper-folding scheme with associated paper surface S and scar G . Suppose that the hypothesis of Theorem 59 is satisfied at some point $q \in G$ other than a non-isolated singularity (as explained in Remark 60, the hypothesis is necessarily satisfied when q is a planar point or regular vertex). Then for any sufficiently small q -planar radius r , $D = \text{Int}(D(q; r))$ is a conformal disk, and by the Uniformization Theorem there is a conformal isomorphism $u: D \rightarrow \mathbb{D}$ to the unit disk $\mathbb{D} \subset \mathbb{C}$. In this section a modulus of continuity for u is obtained.

Definition 79 (Modulus of continuity). Let $\rho: [0, \delta) \rightarrow [0, \infty)$, for some $\delta > 0$, be a continuous and strictly increasing function with $\rho(0) = 0$. A function $f: (X, d_X) \rightarrow (Y, d_Y)$ between metric spaces has *modulus of continuity* ρ at $x_0 \in X$ if, for every $x \in X$ with $d_X(x_0, x) < \delta$,

$$d_Y(f(x_0), f(x)) \leq \rho(d_X(x_0, x)).$$

f is said to have *modulus of continuity* ρ if the inequality above holds for every $x_0 \in X$.

Remark 80. If \mathcal{F} is a family of functions all of whose members have the same modulus of continuity, then \mathcal{F} is *uniformly equicontinuous*: for all $\varepsilon > 0$ there exists $\eta > 0$ such that if two points are at distance less than η then their images under any function in \mathcal{F} (whose domain contains the two points) are at distance less than ε . Notice that there is no requirement for the domains of the functions in \mathcal{F} to coincide: indeed, in the extreme case they may have empty intersection.

6.1. Modulus of continuity at a point. Here a local version of the main result of this section is presented: it provides an illustration of the ideas of the proof without the technical details which obscure the argument in the global case.

Let (P, \mathcal{P}) be a surface paper-folding scheme, q be an isolated singularity in the scar G , and suppose that the hypothesis of Theorem 59 is satisfied at q , so that the complex structure on $S \setminus \mathcal{V}^s$ extends uniquely across q . It will be shown how it is possible to obtain a modulus of continuity for a uniformizing map u from a disk neighborhood of q to the unit disk in the complex plane.

Pick a q -planar radius $r_0 \in (0, \bar{r})$ small enough that $D := \text{Int}(D(q; r_0))$ contains no singularities other than q . Let $u: D \rightarrow \mathbb{D}$ be a conformal uniformizing chart, where \mathbb{D} is the open unit disk in the complex plane: normalize u so that $u(q) = 0$.

It is shown in Lemma 86 below that there are numbers $\delta > 0$ and $A > 0$, depending only on the polygon constants of P , such that

$$B_S(q; r) \subset D(q; Ar) \quad \text{for all } r \leq \delta.$$

Pick any $x \in D$ with $d_S(q, x) < \min\{\delta, r_0/A\}$, and write $R := A d_S(q, x) < r_0$, so that $x \in D(q; R)$. Assume that R is a q -planar radius (if not, increase it by an arbitrarily small amount). Then $\text{Ann}(q; R, r_0)$ separates q and x from $\partial D(q; r_0)$, and hence $u(\text{Ann}(q; R, r_0))$ separates $0 = u(q)$ and $u(x)$ from the unit circle in \mathbb{C} . It follows from (17) and the Grötzsch annulus theorem (Theorem 25) that

$$\begin{aligned} \int_R^{r_0} \iota(q; t) \, dt &\leq \text{mod } \text{Ann}(q; R, r_0) \\ &= \text{mod } u(\text{Ann}(q; R, r_0)) \\ &\leq \frac{1}{2\pi} \ln \frac{4}{|u(x)|}. \end{aligned}$$

Hence, recalling that $R = A d_S(q, x)$ (or R is greater than $A d_S(q, x)$ by an arbitrarily small amount in the case where R is not q -planar),

$$(18) \quad |u(x)| \leq \frac{4}{\exp \left(2\pi \int_{A d_S(q, x)}^{r_0} \iota(q; t) \, dt \right)},$$

which is the desired modulus of continuity.

Remark 81. Since the projection $\pi: P \rightarrow S$ is distance non-increasing, a modulus of continuity ρ_q for u at q is also a modulus of continuity for the composition $\phi := u \circ \pi$ at the points of $\pi^{-1}(q)$: if $q = \pi(\tilde{q})$ and $x = \pi(\tilde{x})$, then

$$\begin{aligned} |\phi(\tilde{q}) - \phi(\tilde{x})| &= |u(q) - u(x)| \\ &\leq \rho_q(d_S(q, x)) \\ &\leq \rho_q(d_P(\tilde{q}, \tilde{x})), \end{aligned}$$

since $d_S(q, x) \leq d_P(\tilde{q}, \tilde{x})$ and ρ_q is increasing.

Example 82. Consider Examples 35 and 63 and suppose as in the second case of Example 63 that the fold lengths a_n satisfy $a_n \asymp 1/\lambda^n$ for some $\lambda > 1$. As shown in that example, the hypothesis of Theorem 59 holds at the unique singularity $q_0 \in G$, and hence the paper surface S is conformally isomorphic to the Riemann sphere. Choose a uniformizing map $u: S \rightarrow \widehat{\mathbb{C}}$ with $u(q_0) = 0$, and let $\phi: P \rightarrow \widehat{\mathbb{C}}$ be the composition $\phi := u \circ \pi$. Thus $\phi(0) = 0$: an explicit modulus of continuity for ϕ at 0 will now be found.

From the calculations of Example 63, a paper-folding goodness function at q_0 can be taken to be

$$\iota(q_0; r) = \frac{1}{Cr \ln \frac{1}{r}},$$

for some positive constant C , so that

$$\int_R^{r_0} \iota(q_0; t) dt = \frac{1}{C} \left(\ln \ln \frac{1}{R} - \ln \ln \frac{1}{r_0} \right).$$

Choose r_0 small enough that $u(D(q; r_0)) \subset \mathbb{D}$. Substitution in (18) yields, for \tilde{x} close to 0 (how close depends on the choice of uniformizing map u)

$$|\phi(\tilde{x})| \leq 4 \left[\frac{\ln(1/r_0)}{\ln(1/(A|\tilde{x}|))} \right]^{2\pi/C}.$$

6.2. Global modulus of continuity. In the remainder of Section 6, the following assumptions will be made:

- a) (P, \mathcal{P}) is a plain paper-folding scheme;
- b) there are only finitely many singular points in the scar G ; and
- c) at each of these singular points, the hypothesis of Theorem 59 is satisfied.

It will be shown how to obtain a global modulus of continuity for a suitably normalized uniformizing map $u: S \rightarrow \widehat{\mathbb{C}}$ from the paper sphere S to the Riemann sphere (and hence also for the composition $\phi = u \circ \pi: P \rightarrow \widehat{\mathbb{C}}$).

Let $\iota(q; r)$ be a paper-folding goodness function for (P, \mathcal{P}) . Then

$$(19) \quad \int_0^\infty \iota(q; t) dt = \infty$$

for every $q \in G$: at singular points this is the assumption c) above, while at other points the integral diverges as explained in Remark 60.

Recall (Definition 68) that the polygon constants of P are the numbers \bar{h} , \bar{r} , and $|\partial P|$: here \bar{h} is the height of the collaring of Section 5.3, and, since the paper-folding is plain, $\bar{r} > 0$ can be chosen arbitrarily. A sensible choice is $\bar{r} = \bar{h}$, which maximises the constant M in the goodness function ι , with $M = 1/5$.

For each $h \in (0, \bar{h}]$, write

$$P_h := P \setminus \tilde{Q}(h),$$

an open disk in $P \subset \mathbb{C}$ which is the complement of the collaring of height h .

To fix a uniformizing map $u: S \rightarrow \widehat{\mathbb{C}}$, pick points $\tilde{p}_0 \in \partial P$ and $\tilde{p}_\infty \in P_h$. Set $p_0 := \pi(\tilde{p}_0) \in G$ and $p_\infty := \pi(\tilde{p}_\infty) \in S \setminus Q(\bar{h})$, and define $u: S \rightarrow \widehat{\mathbb{C}}$ to be the isomorphism with the following normalization:

- $u(p_0) = 0$;
- $u(p_\infty) = \infty$; and
- the reciprocal $\Phi = 1/\phi$ of the composition $\phi := u \circ \pi: P \rightarrow \widehat{\mathbb{C}}$ satisfies $\Phi'(\tilde{p}_\infty) = 1$.

Observe that ϕ is injective and meromorphic in $\text{Int}(P)$ with a simple pole at \tilde{p}_∞ .

Lemma 83. *There is a constant $R = R(\bar{h}) > 0$, depending only on \bar{h} , such that*

$$\phi\left(\tilde{Q}(\bar{h}/2)\right) \subset \mathbb{D}_R := \{z \in \mathbb{C} : |z| < R\}$$

(or, equivalently, $u(Q(\bar{h}/2)) \subset \mathbb{D}_R$).

Proof. Since it lies in $P_{\bar{h}}$, the point \tilde{p}_∞ is distance at least $\bar{h}/2$ from any point of $\tilde{Q}(\bar{h}/2)$: that is, $\tilde{Q}(\bar{h}/2) \subset P \setminus B_{\mathbb{C}}(\tilde{p}_\infty, \bar{h}/2)$. Therefore

$$\Phi\left(\tilde{Q}(\bar{h}/2)\right) \subset \Phi\left(P \setminus B_{\mathbb{C}}(\tilde{p}_\infty; \bar{h}/2)\right) = \widehat{\mathbb{C}} \setminus \Phi\left(B_{\mathbb{C}}(\tilde{p}_\infty; \bar{h}/2)\right),$$

where $\Phi = 1/\phi$. Since Φ is univalent in $B_{\mathbb{C}}(\tilde{p}_\infty; \bar{h}/2)$ with $\Phi(\tilde{p}_\infty) = 0$ and $\Phi'(\tilde{p}_\infty) = 1$, Koebe's one-quarter theorem gives a radius $s(\bar{h}) > 0$ such that $\Phi\left(B_{\mathbb{C}}(\tilde{p}_\infty; \bar{h}/2)\right) \supset \overline{\mathbb{D}}_{s(\bar{h})}$, and the result follows with $R(\bar{h}) = 1/s(\bar{h})$. \square

The main theorem which will be proved in this section is:

Theorem 84. *Let (P, \mathcal{P}) be a plain paper-folding scheme with only finitely many singular points in its scar G , at each of which the conditions of Theorem 59 hold. Then the uniformizing map $\phi: P \rightarrow \widehat{\mathbb{C}}$ has a modulus of continuity $\bar{\rho}$, with respect to the Euclidean metric on P and the spherical metric on $\widehat{\mathbb{C}}$, which depends only on the polygon constants of P and on the paper-folding goodness function $\iota: G \times (0, \bar{r}) \rightarrow [0, \infty)$.*

Recall that the spherical metric $d_{\widehat{\mathbb{C}}}$ on $\widehat{\mathbb{C}}$ is defined by

$$d_{\widehat{\mathbb{C}}}(w, z) = \inf_{\gamma \in \Gamma} \left(2 \int_{\gamma} \frac{|dz|}{1 + |z|^2} \right),$$

where Γ is the set of paths in $\widehat{\mathbb{C}}$ from w to z . The only properties of the spherical metric which will be used here are: $d_{\widehat{\mathbb{C}}}(1/w, 1/z) = d_{\widehat{\mathbb{C}}}(w, z)$ for all $w, z \in \widehat{\mathbb{C}}$; and $d_{\widehat{\mathbb{C}}}(w, z) \leq 2|w - z|$ for all $w, z \in \mathbb{C}$.

6.3. Two technical lemmas. Recall from Section 5.3.2 that $\tilde{\psi}_{\bar{h}}: \tilde{Q}(\bar{h}) \rightarrow \partial P$ is the retraction obtained by sliding points along leaves of $\tilde{\text{Ver}}$: for economy of notation, this retraction will henceforth be denoted $\tilde{\psi}$, and similarly ψ rather than $\psi_{\bar{h}}$ will be used to denote the retraction $\pi \circ \tilde{\psi} \circ \pi^{-1}: Q(\bar{h}) \rightarrow G$. The first lemma in this section essentially says that ψ is Lipschitz with constant determined by the polygon constants.

Lemma 85. *Let α be a rectifiable path in $Q(\bar{h})$. Then*

$$|\psi \circ \alpha|_G \leq \frac{|\partial P|}{\bar{h}} |\alpha|_S.$$

Sketch proof. Lift α to a path in $\tilde{Q}(\bar{h})$, and consider a point $\tilde{\gamma}(t, h)$ of α lying in a trapezoid \tilde{Q}_i . As in the proof of Lemma 70, extend the vertical sides of \tilde{Q}_i to form a triangle T of height $H \geq 2\bar{h}$. The action of $\tilde{\psi}$ in \tilde{Q}_i is projection from the apex of T onto the base of \tilde{Q}_i .

An infinitesimal arc through $\tilde{\gamma}(t, h)$ is maximally stretched by $\tilde{\psi}$ when it is perpendicular to the line segment from $\tilde{\gamma}(t, h)$ to the apex of T , in which case it is stretched by a factor $\frac{H}{(H-h)\sin\theta(t)}$. The result follows using (12) and $H \geq 2h$. \square

The following lemma relates metric balls $B_S(q, \delta)$ in S to the disks $D(\psi(q); \mu)$ for q in a thinner collar $Q(\delta) = \pi \circ \tilde{Q}(\delta)$ (see Figure 9).

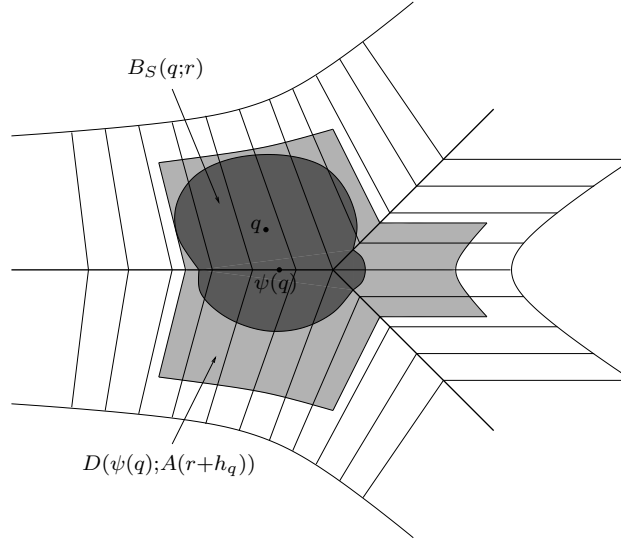


FIGURE 9. The disk $D(\psi(q); A(r + h_q))$ and the ball $B_S(q; r)$.

Lemma 86. *Let*

$$(20) \quad \delta = \frac{1}{4} \min \left\{ \bar{h}, \bar{r}, \frac{2\bar{h}\bar{r}}{|\partial P|} \right\} \quad \text{and} \quad A = \frac{\bar{r}}{2\delta} = 2 \max \left\{ \frac{\bar{r}}{\bar{h}}, 1, \frac{|\partial P|}{2\bar{h}} \right\}.$$

Then for every $q \in Q(\delta)$ and every $r \in [0, \delta]$,

$$\overline{B}_S(q; r) \subset D(\psi(q); A(r + h_q)),$$

where $h_q \in [0, \delta]$ is the height of q (i.e. $q \in \text{hor}(h_q)$).

Remark 87. Notice that $A(r + h_q) \leq A(\delta + \delta) = \bar{r}$, so that $D(\psi(q); A(r + h_q))$ is defined (even when $A(r + h_q)$ is not a $\psi(q)$ -planar radius, in which case it may not be a disk).

Proof of Lemma 86. Observe first that the d_S -distance from q to the boundary $\text{hor}(\bar{h})$ of $Q = Q(\bar{h})$ is at least $\bar{h} - h_q \geq \bar{h} - \delta \geq 3\bar{h}/4 > \delta \geq r$, so that $\bar{B}_S(q; r) \subset Q(\bar{h})$.

Write $K = A(r + h_q)$. Since $D(\psi(q); K) = \psi_{h(K)}^{-1}(\bar{B}(q; K))$ where $h(K) = \bar{h}K/2\bar{r}$ (Definition 65), it is required to show that every point $x \in S$ with $d_S(q, x) \leq r$ has height $h_x \leq \bar{h}K/2\bar{r}$ and satisfies $d_G(\psi(q), \psi(x)) \leq K$.

The former property is immediate since $h_x \leq r + h_q = K/A \leq \bar{h}K/2\bar{r}$. For the latter, since d_S is strictly intrinsic (Theorem 56b)), there is a path in S from q to x of length $d_S(q, x)$, and this path must lie in $Q(\bar{h})$ since the distance from q to $\text{hor}(\bar{h})$ is greater than r . The image under ψ of this path is a path in G from $\psi(q)$ to $\psi(x)$, whose length is no more than $\frac{|\partial P|}{\bar{h}} d_S(q, x)$ by Lemma 85. Hence

$$d_G(\psi(q), \psi(x)) \leq \frac{|\partial P|}{\bar{h}} r \leq A(r + h_q) = K$$

as required. \square

6.4. Modulus of continuity in a collar. Let δ and A be given by (20), and $R = R(\bar{h}) > 0$ be the constant given by Lemma 83. Define functions $\mu: Q(\delta) \times [0, \delta) \rightarrow [0, \bar{r})$ and $\rho: Q(\delta) \times [0, \delta) \rightarrow [0, \infty)$ by

$$\mu(q, t) := A(t + h_q)$$

(so $\mu(q, t) < 2A\delta = \bar{r}$) and

$$(21) \quad \rho_q(t) = \rho(q, t) := \begin{cases} 0, & \text{if } t = 0; \\ \frac{8Rt}{h_q \cdot \exp\left(2\pi \int_{\mu(q, h_q)}^{\bar{r}} \iota(\psi(q); s) ds\right)}, & \text{if } 0 < t \leq h_q; \\ \frac{8R}{\exp\left(2\pi \int_{\mu(q, t)}^{\bar{r}} \iota(\psi(q); s) ds\right)}, & \text{if } h_q \leq t \leq \delta. \end{cases}$$

It will be shown in this section that, for every $\tilde{q} \in \tilde{Q}(\delta)$, $\rho_{\pi(\tilde{q})}$ is a modulus of continuity for $\phi: P \rightarrow \hat{\mathbb{C}}$ at \tilde{q} .

Remark 88. For each $q \in Q(\delta)$, $\rho_q: [0, \delta) \rightarrow [0, \infty)$ is a modulus of continuity in the sense of Definition 79:

a) It is continuous in $(0, \delta)$ and strictly increasing since:

- when $t \in (0, h_q]$, $\rho_q(t)$ is proportional to t ;

- when $t \in (h_q, \delta)$, $\rho_q(t)$ is inversely proportional to $\exp \left(2\pi \int_{A(t+h_q)}^{\bar{r}} \iota(\psi(q); s) \, ds \right)$, which depends continuously on t , and is strictly decreasing since $A(t+h_q)$ is strictly increasing and ι is positive except on a set of measure zero; and
- when $t = h_q > 0$, the two expressions for ρ_q clearly agree.

b) It is continuous at 0 since

- If $q \in G$ (i.e. $h_q = 0$), the second expression in (21) is used, and the integral in its denominator diverges as $t \rightarrow 0$ by assumption; and
- If $q \notin G$ (i.e. $h_q > 0$), the first expression in (21) is used for t sufficiently small, which clearly converges to 0 as $t \rightarrow 0$.

The (joint) continuity of $\rho: Q(\delta) \times [0, \delta) \rightarrow [0, \infty)$ will be proved in Lemma 92 below.

Example 89. For the simplest possible example, suppose that $q \in G$ is a planar point, and that \bar{r} is small enough that $m(q; r) = 4r$ and $n(q; r) = 2$ for all $r \leq \bar{r}$: hence $\iota(q; r) = M/6r$. Then (21) gives a modulus of continuity

$$\begin{aligned} \rho_q(t) &= \frac{8R}{\exp \left(\int_{At}^{\bar{r}} \frac{\pi M}{3s} \, ds \right)} \\ &= 8R \left(\frac{A}{\bar{r}} \right)^{\frac{\pi M}{3}} t^{\frac{\pi M}{3}}. \end{aligned}$$

Since $M \leq 1/5$ (Definition 77), the exponent $\pi M/3$ lies in $(0, 1)$. This modulus of continuity is of course not optimal, since the uniformizing map is Lipschitz near planar points.

Lemma 90. *If $\tilde{q} \in \tilde{Q}(\delta)$, then $\phi: P \rightarrow \hat{\mathbb{C}}$ has modulus of continuity $\rho_{\pi(\tilde{q})}$ at \tilde{q} . That is, if $\tilde{q} \in \tilde{Q}(\delta)$ and $\tilde{x} \in P$ with $d_P(\tilde{q}, \tilde{x}) < \delta$, then*

$$|\phi(\tilde{q}) - \phi(\tilde{x})| \leq \rho_{\pi(\tilde{q})}(d_P(\tilde{q}, \tilde{x})).$$

Proof. Notice that $\phi(\tilde{q})$ and $\phi(\tilde{x})$ lie in \mathbb{C} (since $\phi^{-1}(\infty) = \tilde{p}_\infty \notin \tilde{Q}(\bar{h})$), and so $|\phi(\tilde{q}) - \phi(\tilde{x})|$ makes sense.

As explained in Remark 81, a modulus of continuity for $u: S \rightarrow \hat{\mathbb{C}}$ is also a modulus of continuity for $\phi = u \circ \pi$, and it therefore suffices to obtain the former. So let $q \in Q(\delta)$, and let $x \in S$ be such that

$$t := d_S(q, x) < \delta.$$

It is required to show that $|u(q) - u(x)| \leq \rho_q(t)$.

Case 1: $h_q \leq t$

$x \in D(\psi(q); \mu(q, t))$ by Lemma 86. Assuming at first that $\mu(q, t)$ is a $\psi(q)$ -planar radius, this means that the annular region $\text{Ann}(\psi(q); \mu(q, t), \bar{r})$ separates q and x from $S \setminus Q(\bar{h}/2)$. Hence, by Lemma 83, the image $u(\text{Ann}(\psi(q); \mu(q, t), \bar{r}))$ separates $u(q)$ and $u(x)$ from the circle $\{|z| = R\}$.

The modulus of continuity can thus be obtained from the Grötzsch annulus theorem (Theorem 25) just as in Section 6.1:

$$\begin{aligned}
\int_{\mu(q,t)}^{\bar{r}} \iota(\psi(q); s) \, ds &\leq \text{mod Ann}(\psi(q); \mu(q, t), \bar{r}) \\
&= \text{mod } u(\text{Ann}(\psi(q); \mu(q, t), \bar{r})) \\
&\leq \text{mod Gr} \left(\left| \frac{u(q) - u(x)}{R - \frac{u(x)\overline{u(q)}}{R}} \right| \right) \\
&\leq \text{mod Gr} \left(\frac{|u(q) - u(x)|}{2R} \right) \\
&\leq \frac{1}{2\pi} \ln \frac{8R}{|u(q) - u(x)|}
\end{aligned}$$

(where the Möbius transformation $z \mapsto \frac{R(z-u(q))}{R^2-zu(q)}$ has been used to move $u(q)$ to 0 and the circle $|z| = R$ to the circle $|z| = 1$), giving

$$|u(q) - u(x)| \leq \frac{8R}{\exp \left(2\pi \int_{\mu(q,t)}^{\bar{r}} \iota(\psi(q); s) \, ds \right)}$$

as required.

If $\mu(q; t)$ is not $\psi(q)$ -planar, then increasing it by an arbitrarily small amount to a planar radius gives the result.

Case 2: $0 < t \leq h_q$ Since $x \in D(\psi(q); \mu(q, t)) \subset D(\psi(q); \mu(q, h_q))$, the annular region $\text{Ann}(\psi(q); \mu(q, h_q), \bar{r})$ separates q and x from $S \setminus Q(\bar{h}/2)$ (if $\mu(q, h_q)$ is not a planar radius, then perturb as in Case 1). Moreover, since $\overline{B}_S(q; h_q) \subset D(\psi(q); \mu(q, h_q))$ by Lemma 86, the annular region $B_S(q; h_q) \setminus \overline{B}_S(q; t)$ is nested inside the first annulus and also separates q and x from $S \setminus Q(\bar{h}/2)$. Arguing as in Case 1, it follows that

$$\text{mod} (B_S(q; h_q) \setminus \overline{B}_S(q; t)) + \text{mod Ann}(\psi(q); \mu(q, h_q), \bar{r}) \leq \frac{1}{2\pi} \ln \frac{8R}{|u(q) - u(x)|}.$$

Since $B_S(q; h_q)$ is isometric to a plane metric ball,

$$\text{mod} (B_S(q; h_q) \setminus \overline{B}_S(q; t)) = \frac{1}{2\pi} \ln \frac{h_q}{t},$$

and hence

$$\frac{1}{2\pi} \ln \frac{h_q}{t} + \int_{\mu(q, h_q)}^{\bar{r}} \iota(\psi(q); s) \, ds \leq \frac{1}{2\pi} \ln \frac{8R}{|u(q) - u(x)|}$$

so that

$$|u(q) - u(x)| \leq \frac{8Rt}{h_q \cdot \exp \left(2\pi \int_{\mu(q, h_q)}^{\bar{r}} \iota(\psi(q), s) \, ds \right)}$$

as required. \square

In order to use Lemma 90 to construct a global modulus of continuity in $Q(\delta)$, the maximum over the functions ρ_q will be taken. To show that this maximum is itself continuous it is necessary to prove that $\rho: Q(\delta) \times [0, \delta] \rightarrow [0, \infty)$ is continuous: this is the aim of the remainder of this section.

Lemma 91. *Let (P, \mathcal{P}) be a plain paper-folding scheme with only finitely many singular points in its scar G , at each of which the conditions of Theorem 59 hold. Define $I: G \times (0, \bar{r}) \rightarrow (0, \infty)$ by*

$$I(q, r) = \int_r^{\bar{r}} \iota(q; s) \, ds.$$

Then

- a) I is continuous, and
- b) For all $K > 0$, there is some $\eta > 0$ such that $I(q, r) > K$ for all $q \in G$ and all $r \in (0, \eta)$.

Proof. Recall that

$$I(q, r) = \int_r^{\bar{r}} \iota(q; s) \, ds = M \int_r^{\bar{r}} \frac{ds}{m(q; s) + s \cdot n(q; s)},$$

where $m(q; s) = m_G(B_G(q; s))$.

- a) Let $(q_0; r_0) \in G \times (0, \bar{r})$: it will be shown that I is continuous at $(q_0; r_0)$. Now for $(q; r) \in G \times (0, \bar{r})$,

$$I(q, r) - I(q_0, r_0) = \int_{r_0}^{\bar{r}} \iota(q; s) \, ds - \int_{r_0}^{\bar{r}} \iota(q_0; s) \, ds + \int_r^{r_0} \iota(q; s) \, ds.$$

The final integral converges to zero as $(q, r) \rightarrow (q_0, r_0)$, since $\iota(q; s)$ is bounded above by M/r_0 for $s \geq r_0/2$. Hence it suffices to prove that, for all $\varepsilon > 0$,

$$\left| \int_{r_0}^{\bar{r}} (\iota(q; s) - \iota(q_0; s)) \, ds \right| = M \left| \int_{r_0}^{\bar{r}} \left(\frac{1}{m(q; s) + s \cdot n(q; s)} - \frac{1}{m(q_0; s) + s \cdot n(q_0; s)} \right) \right| < \varepsilon$$

provided $d_G(q, q_0)$ is sufficiently small.

Because the absolute value of the integrand is bounded above by M/r_0 , a small open interval of radius δ_1 can be excised from the range of integration about each of the finitely many values of s for which there is a singularity at distance s from q_0 , without changing the integral by more than $\varepsilon/3$. Again, since there are now only finitely many values of s in the range of integration for which there is a vertex at distance s from q_0 , further open intervals of radius $\delta_2 \leq \delta_1$ can be excised about each of these values without changing the integral by more than an additional $\varepsilon/3$.

Now if $d_G(q, q_0) < \delta_2/2$ then $n(q_0; s) = n(q; s)$ in each of the remaining intervals of integration. Moreover, $m(q; s)$ is continuous in both variables for $q \in B_G(q_0; \delta_2/2)$ and s in a single interval of integration, and the result follows.

b) Let $q_0 \in G$. Since (8) holds at q_0 , there is some $\varepsilon > 0$ such that

$$M \int_{\varepsilon}^{\bar{r}} \frac{ds}{m(q_0; s) + s \cdot n(q_0; s)} > 3K.$$

The integrand is bounded above by M/ε so, as in part a), there is a subset L of $[\varepsilon, \bar{r}]$ and a number $\delta > 0$ such that:

- L consists of finitely many intervals;
- $(d_G(q_0, q^*) - \delta, d_G(q_0, q^*) + \delta)$ is disjoint from L for any non-planar point q^* of G ; and
-

$$M \int_L \frac{ds}{m(q_0; s) + s \cdot n(q_0; s)} > 2K.$$

Now if $s \in L$ then $n(q_0; s) = n(q; s)$ for $d_G(q, q_0) < \delta/2$; and m is continuous in $B_G(q_0; \delta/2) \times L$. There is therefore some $\delta' \in (0, \delta/2)$ such that

$$M \int_L \frac{ds}{m(q; s) + s \cdot n(q; s)} > K$$

provided that $d_G(q, q_0) < \delta'$. Hence

$$I(q, r) > \int_{\varepsilon}^{\bar{r}} \iota(q; s) ds \geq \int_L \iota(q; s) ds > K$$

for all q with $d_G(q, q_0) < \delta'$ and all $r \in (0, \varepsilon)$. The result follows by compactness of G . □

Lemma 92. *Let (P, \mathcal{P}) be a plain paper-folding scheme with only finitely many singular points in its scar G , at each of which the conditions of Theorem 59 hold. Then the function $\rho: Q(\delta) \times [0, \delta) \rightarrow [0, \infty)$ of (21) is continuous.*

Proof. Observe that if $\xi: Q(\delta) \times (0, \delta) \rightarrow [1, \infty)$ is defined by

$$\xi(q, t) = \max \left(1, \frac{h_q}{t} \right),$$

then (21) can be written

$$\rho(q, t) = \begin{cases} 0, & \text{if } t = 0; \\ \frac{8R}{\xi(q, t) \cdot \exp \left(2\pi \int_{\mu(q, t\xi(q, t))}^{\bar{r}} \iota(\psi(q); s) ds \right)}, & \text{if } t > 0. \end{cases}$$

Since ξ , μ , and ψ are continuous, the continuity of ρ at points (q_0, t_0) with $t_0 > 0$ follows from Lemma 91a).

It remains to prove that ρ is continuous at all points $(q_0, 0)$ with $q_0 \in Q(\delta)$: that is, that $\rho(q, t) \rightarrow 0$ as $(q, t) \rightarrow (q_0, 0)$ with $t > 0$.

If $h_{q_0} > 0$ then $\xi(q, t) = h_q/t$ for all (q, t) close to $(q_0, 0)$, so that $\xi(q, t) \rightarrow \infty$, and hence $\rho(q, t) \rightarrow 0$, as $(q, t) \rightarrow (q_0, 0)$.

If $h_{q_0} = 0$ (i.e. $q_0 \in G$) then, since $\xi(q, t) \geq 1$ for all (q, t) , the result is immediate from Lemma 91b) and the continuity of ψ . \square

6.5. Modulus of continuity in the interior. In Section 6.4, a modulus of continuity for ϕ was obtained at points $\tilde{q} \in \tilde{Q}(\delta)$, where δ is given by (20). In this section, a modulus of continuity is derived at points \tilde{q} in the complement $P_{\delta/2}$ of $\tilde{Q}(\delta/2)$. These overlapping moduli of continuity will then be glued together in Section 6.6 to give the required global modulus.

Lemma 93. *Let $\tilde{q}, \tilde{x} \in P_{\delta/2}$. Then*

$$d_{\widehat{\mathbb{C}}}(\phi(\tilde{q}), \phi(\tilde{x})) \leq \kappa d_P(\tilde{q}, \tilde{x}),$$

where

$$(22) \quad \kappa = 2 \exp \left(\frac{16 \operatorname{diam}_{P_{\delta/2}} P_{\delta/2}}{\delta} \right) \leq 2 \exp \left(\frac{32 |\partial P|}{\delta} \right)$$

(recall that $d_{\widehat{\mathbb{C}}}$ denotes the spherical metric on $\widehat{\mathbb{C}}$). That is, ϕ is Lipschitz in $P_{\delta/2}$ with a constant which depends only on the polygon constants of P .

Proof. Let $\Phi: \operatorname{Int}(P) \rightarrow \mathbb{C}$ be defined by $\Phi := 1/\phi$. Then Φ is univalent with $\Phi(\tilde{p}_\infty) = 0$ and $\Phi'(\tilde{p}_\infty) = 1$ by the choice of normalization of ϕ . It follows from Theorem 26 that Φ is Lipschitz in $P_{\delta/2}$ with constant $\kappa/2$, where κ is as in the statement of the lemma. Then

$$\begin{aligned} d_{\widehat{\mathbb{C}}}(\phi(\tilde{q}), \phi(\tilde{x})) &= d_{\widehat{\mathbb{C}}}(\Phi(\tilde{q}), \Phi(\tilde{x})) \\ &\leq 2 |\Phi(\tilde{q}) - \Phi(\tilde{x})| \\ &\leq \kappa d_P(\tilde{q}, \tilde{x}) \end{aligned}$$

as required. Finally, observe that

$$\operatorname{diam}_{P_{\delta/2}} P_{\delta/2} \leq 2 |\partial P|$$

since the path between any two points of $P_{\delta/2}$ obtained by connecting each with a horizontal arc to the boundary and then joining the endpoints of these arcs with the shorter boundary arc between them has length bounded above by $\frac{1}{2}|\partial P| + \frac{1}{2}|\partial P| + \frac{1}{2}|\partial P_{\delta/2}| \leq 2|\partial P|$ by (11). \square

6.6. Proof of Theorem 84. Define $\hat{\rho}: [0, \delta) \rightarrow (0, \infty)$ by

$$\hat{\rho}(t) := 2 \max_{q \in Q(\delta)} \rho(q, t),$$

which is well defined since ρ is continuous and $Q(\delta)$ is compact. Then $\hat{\rho}$ is a continuous strictly increasing function with $\hat{\rho}(0) = 0$ (since each ρ_q has these properties and ρ is continuous), and ϕ has modulus of continuity $\hat{\rho}$ on $\tilde{Q}(\delta)$ with respect to the Euclidean metric on $\tilde{Q}(\delta)$ and the spherical metric on $\hat{\mathbb{C}}$ (the factor 2 arises from the translation from the Euclidean to the spherical metric).

On the other hand, ϕ is κ -Lipschitz in $P_{\delta/2}$ by Lemma 93, where $\kappa = \kappa(\delta, |\partial P|)$ is given by (22). Hence $\bar{\rho}: [0, \delta) \rightarrow [0, \infty)$ defined by

$$\bar{\rho}(t) := \max \{ \hat{\rho}(t), \kappa t \}$$

is the desired modulus of continuity.

7. A DYNAMICAL APPLICATION: CONVERGENCE TO THE TIGHT HORSESHOE

7.1. Introduction. This section contains an extended example. It illustrates that it is practicable to estimate the modulus of continuity provided by (21) uniformly across a family of paper-folding schemes, and thereby to construct limits which are of interest in dynamical systems theory. The example is the simplest non-trivial one known to the authors, but the context in which it arises, which is described below, provides a wealth of similar cases. In fact, similar arguments construct an uncountable collection of limiting maps in the family of unimodal generalized pseudo-Anosov maps defined in [6] as will be shown elsewhere.

The family of *tent maps* $T_\lambda: [0, 1] \rightarrow [0, 1]$, where $\lambda \in (1, 2]$, is defined by

$$T_\lambda(x) = \begin{cases} \lambda(x - 1) + 2 & \text{if } x \leq 1 - \frac{1}{\lambda}, \\ \lambda(1 - x) & \text{if } x \geq 1 - \frac{1}{\lambda}. \end{cases}$$

Thus $T_\lambda(0) = 2 - \lambda$, $T_\lambda(1 - \frac{1}{\lambda}) = 1$, $T_\lambda(1) = 0$, and T_λ has constant slope λ on $[0, 1 - \frac{1}{\lambda}]$ and constant slope $-\lambda$ on $[1 - \frac{1}{\lambda}, 1]$ (see Figure 10).

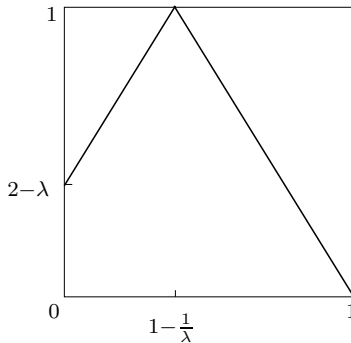


FIGURE 10. Graph of the tent map T_λ

Tent maps have been widely studied in the field of unimodal dynamics, not least because every unimodal map $f: [0, 1] \rightarrow [0, 1]$ with positive topological entropy $h = h(f)$ is semi-conjugate to the tent map of slope e^h [13]. In this section, only tent maps with slopes $\lambda \in (\sqrt{2}, 2]$ will be considered.

There is a countable dense subset $\Lambda \subset (\sqrt{2}, 2]$ of parameters λ for which 1 is either a *periodic* ($T_\lambda^n(1) = 1$ for some $n \geq 1$) or a *preperiodic* ($T_\lambda^k(1)$ is periodic for some $k \geq 0$) point of T_λ . In [6] the authors constructed, for each $\lambda \in \Lambda$, a complexification $f_\lambda: S_\lambda \rightarrow S_\lambda$ of T_λ as a quasi-conformal automorphism of a complex sphere S_λ . Naturally, one would like to be able to define such complexifications f_λ for all $\lambda \in (\sqrt{2}, 2]$ by taking limits through Λ , and this was one of the motivations for the work described in the current paper. The above spheres S_λ are constructed as paper spheres, and the first step in taking limits is to identify each S_λ with the Riemann sphere $\widehat{\mathbb{C}}$ by a suitable choice of normalization. In order to construct limits, it is necessary to show that the uniformizing maps $u_\lambda: S_\lambda \rightarrow \widehat{\mathbb{C}}$ have a uniform modulus of continuity.

The family $(f_\lambda)_{\lambda \in \Lambda}$ contains a subfamily, parameterized by rationals $m/n \in (0, 1/2)$, for which the paper sphere $S_{m/n} := S_\lambda$ arises from a paper-folding scheme with only finitely many segment pairings: these are the so-called *NBT* examples of [10]. In these cases, $f_{m/n} := f_\lambda$ is a pseudo-Anosov automorphism: it preserves a transverse pair of measured foliations with finitely many singularities, stretching one foliation uniformly by a factor λ and contracting the other uniformly by $1/\lambda$ — the foliations are the projections to S_λ of the horizontal and vertical foliations of the polygon P_λ . For all other $\lambda \in \Lambda$, f_λ is a *generalized pseudo-Anosov*: these are defined in the same way as pseudo-Anosovs, except that their invariant foliations are permitted to have infinitely many singularities, provided that these singularities accumulate in only finitely many points.

The simplest example of a generalized pseudo-Anosov is the *tight horseshoe*, which is the complexification of the *full* tent map T_2 . Its sphere $S = S_2$ of definition is obtained from the plain paper-folding scheme (Σ, \mathcal{P}) , where $\Sigma = [0, 1] \times [0, 1] \subset \mathbb{R}^2$ and the segment pairings \mathcal{P} consist of two folds of length $1/2^i$ for each $i \geq 1$: the top and right sides of Σ are folded in half, and the bottom and left sides are covered by folds of lengths $1/2^i$ for $i \geq 2$, arranged in order of decreasing length from the bottom right and top left vertices respectively (Figure 11). As in Example 63, the conditions of Theorem 59 hold at the unique singular point arising from the identification of the bottom left corner of Σ with all of the fold endpoints, so that S has a unique complex structure induced by the Euclidean structure on Σ . The scar G is an ∞ -od, having two edges of length $1/2^i$ for each $i \geq 1$.

Let $F: \Sigma \rightarrow \Sigma$ be the (discontinuous and non-injective) function defined by

$$F(x, y) = \begin{cases} (2x, \frac{y}{2}) & \text{if } x \leq \frac{1}{2}, \\ (2(1-x), 1 - \frac{y}{2}) & \text{if } x > \frac{1}{2}, \end{cases}$$

so that the first coordinate of $F(x, y)$ is $T_2(x)$, and F contracts by a factor 2 in the vertical direction. The identifications on Σ are precisely those which are needed to make F continuous and injective, and it therefore descends to a homeomorphism $f: S \rightarrow S$. This homeomorphism preserves the

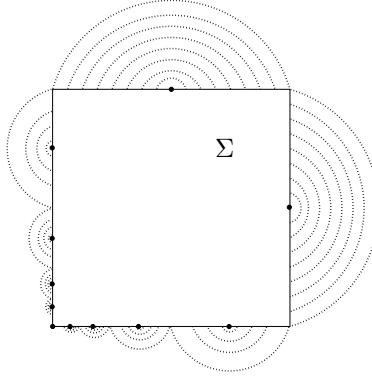


FIGURE 11. The sphere of definition of the tight horseshoe

foliations on S which are the projections of the horizontal and vertical foliations of Σ , stretching the former by a factor 2 and contracting the latter by a factor $1/2$: the foliations have 1-pronged singularities corresponding to the mid-point of each fold, which accumulate on the unique singular point. f is called the tight horseshoe because it can be obtained from Smale's horseshoe map [18] by collapsing the gaps in the non-wandering Cantor set.

The example elaborated in this section treats the convergence of the NBT pseudo-Anosov homeomorphisms $f_{1/n}: S_{1/n} \rightarrow S_{1/n}$ (which will henceforth be denoted $f_n: S_n \rightarrow S_n$) to the tight horseshoe $f: S \rightarrow S$ as $n \rightarrow \infty$: uniformizing maps $u_n: S_n \rightarrow \widehat{\mathbb{C}}$ and $u: S \rightarrow \widehat{\mathbb{C}}$ are chosen so that the homeomorphisms $u_n \circ f_n \circ u_n^{-1}$ of $\widehat{\mathbb{C}}$ converge to $u \circ f \circ u^{-1}$. This will provide a dynamical meaning to the term *tight horseshoe*. It was shown in [10] that the sequence (f_n) effectively exhausts the finite invariant sets of the horseshoe, in the sense that for any finite invariant set A of the horseshoe, there is some N such that, for all $n \geq N$, f_n is the pseudo-Anosov representative in the isotopy class of the horseshoe relative to a finite invariant set $A' \supset A$: thus the tight horseshoe is tight in the sense of the introduction to this article.

The paper-folding schemes (P_n, \mathcal{P}_n) which provide the spheres S_n , and the homeomorphisms $f_n: S_n \rightarrow S_n$ will be described in Section 7.2. This application of the results of the article has been chosen so that an elementary and self-contained description of these spaces and functions can be given without requiring the machinery of train tracks. However, the price for eschewing this machinery is that the constructions are both ad-hoc and involved. In Section 7.3 it is shown that there are uniform polygon constants for the polygons P_n , which makes it possible, in Section 7.4, to obtain a uniform modulus of continuity for the uniformizing maps $\phi_n: P_n \rightarrow \widehat{\mathbb{C}}$ and hence to complete the proof of convergence.

7.2. The paper spheres S_n and the pseudo-Anosov homeomorphisms $f_n: S_n \rightarrow S_n$.

Let $\lambda \in (\sqrt{2}, 2]$. The *itinerary* $k_\lambda(x) \in \{0, C, 1\}^{\mathbb{N}}$ of a point $x \in [0, 1]$ under the tent map T_λ

is defined by

$$k_\lambda(x)_i = \begin{cases} 0 & \text{if } T_\lambda^i(x) < 1 - \frac{1}{\lambda}, \\ C & \text{if } T_\lambda^i(x) = 1 - \frac{1}{\lambda}, \\ 1 & \text{if } T_\lambda^i(x) > 1 - \frac{1}{\lambda}. \end{cases}$$

The *kneading invariant* of T_λ is defined as

$$k(T_\lambda) = k_\lambda(1).$$

The kneading theory of Milnor and Thurston [13] provides a way of understanding the dynamics of T_λ by means of its kneading invariant. In particular, λ itself can be recovered as the reciprocal of the smallest positive root μ of a certain power series. In the case where 1 is a period N point of T_λ , μ is the smallest positive root of the polynomial $\sum_{i=0}^{N-1} \theta_i t^i$ whose coefficients $\theta_i \in \{-1, +1\}$ are given by $\theta_0 = 1$ and

$$\theta_i = \begin{cases} \theta_{i-1} & \text{if } k(T_\lambda)_{i-1} = 0, \\ -\theta_{i-1} & \text{if } k(T_\lambda)_{i-1} = 1 \end{cases}$$

for $1 \leq i < N$.

Let $n \geq 3$. The value λ_n of λ corresponding to the $1/n$ NBT case (which are the ones of interest in this section) is the slope of the tent map with kneading invariant

$$k(T_\lambda) = (10^{n-1}1C)^\infty$$

(n will be fixed throughout most this subsection, and hence the suffix n on λ_n will usually be omitted). Then μ is the smallest positive root of the polynomial

$$f_n(t) = 1 - t - t^2 - \dots - t^n + t^{n+1},$$

and $\lambda = 1/\mu$ is the largest root of the same polynomial. In particular, multiplying through by $(t-1)$, λ satisfies

$$(23) \quad \lambda^{n+2} - 2\lambda^{n+1} + 2\lambda - 1 = 0.$$

(λ_n) is therefore an increasing sequence with $\lambda_n \rightarrow 2$ as $n \rightarrow \infty$.

Since $k(T_\lambda) = (10^{n-1}1C)^\infty$, 1 is a period $n+2$ point of T_λ : write $p_0 = 1$ and $p_i = T_\lambda^i(p_0)$ for $1 \leq i \leq n+1$ so that $T_\lambda(p_{n+1}) = p_0$. The following explicit description of the points p_i is easily verified by induction:

$$(24) \quad p_i = \begin{cases} 1 & \text{if } i = 0, \\ 0 & \text{if } i = 1, \\ \frac{(2-\lambda)(\lambda^{i-1}-1)}{\lambda-1} & \text{if } 2 \leq i \leq n, \\ 1 - \frac{1}{\lambda} & \text{if } i = n+1. \end{cases}$$

Now define a function $F_n: \Sigma \rightarrow \mathbb{R}^2$ by

$$F_n(x, y) = \begin{cases} \left(\lambda_n(x-1) + 2, \frac{y}{\lambda_n} - \frac{1}{\lambda_n^{n+1}+1} \right) & \text{if } x \leq 1 - \frac{1}{\lambda_n}, \\ \left(\lambda_n(1-x), 1 - \frac{y}{\lambda_n} \right) & \text{if } x > 1 - \frac{1}{\lambda_n} \end{cases}$$

so that the first coordinate of $F_n(x, y)$ is $T_{\lambda_n}(x)$, and F_n contracts by a factor $1/\lambda_n$ in the y direction (see Figure 12). Notice that F_n is not injective, and that it is discontinuous across the line $x = 1 - 1/\lambda_n$.

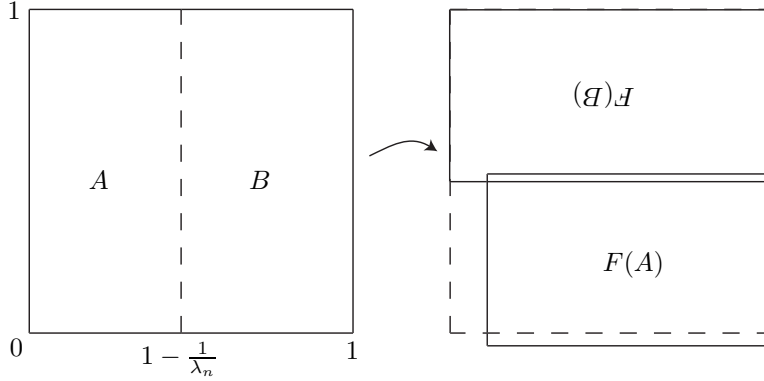


FIGURE 12. The action of $F_n: \Sigma \rightarrow \mathbb{R}^2$

The polygon P_n in the paper-folding scheme (P_n, \mathcal{P}_n) is chosen so that $F_n(P_n) = P_n$ and F_n is injective on its interior. The segment pairings \mathcal{P}_n are then chosen to make the induced action of F_n on the quotient paper sphere S_n continuous and injective. The resulting homeomorphism $f_n: S_n \rightarrow S_n$ will be a pseudo-Anosov map, with $n+2$ one-pronged singularities, and one n -pronged singularity.

P_n has $n+2$ horizontal sides and $n+2$ vertical sides. The vertical sides V_i ($0 \leq i \leq n+1$) are contained in the vertical fibers over the points p_i of the orbit of 1 under the tent map T_λ : they are defined by

$$V_i = \begin{cases} \{1\} \times \left[0, \frac{\lambda^{n+1}}{\lambda^{n+1}+1}\right] & \text{if } i = 0, \\ F_n^i(V_0) & \text{if } 1 \leq i \leq n+1. \end{cases}$$

By (23), the height h of V_0 satisfies

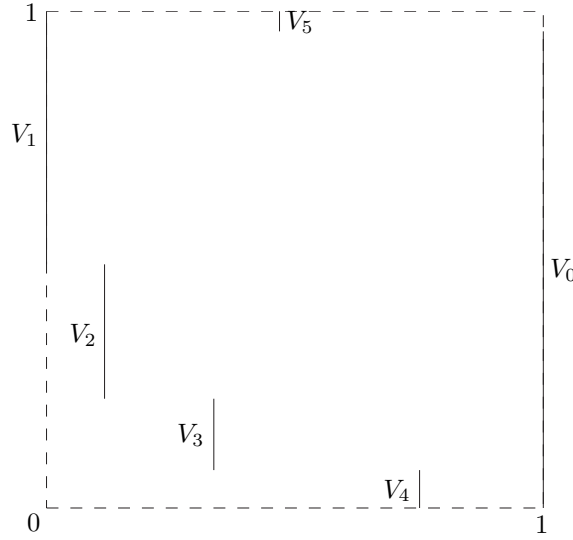
$$h = \frac{\lambda^{n+1}}{\lambda^{n+1}+1} = \frac{2\lambda^{n+1}(\lambda-1)}{\lambda^{n+2}-1},$$

so that the sum $\sum_{i=0}^{n+1} h/\lambda^i$ of the heights of the vertical sides is equal to 2.

The essential point about the configuration of these vertical sides is contained in the following lemma (see Figure 13).

Lemma 94. *Let $\pi_y: \Sigma \rightarrow [0, 1]$ be projection onto the second coordinate. Then*

a) $\pi_y(V_0 \cup V_{n+1}) = [0, 1]$, and $\pi_y(V_0) \cap \pi_y(V_{n+1})$ is a single point.


 FIGURE 13. The vertical sides of P_4

b) $\pi_y(V_1 \cup \dots \cup V_n) = [0, 1]$; and for $1 \leq i < j \leq n$, $\pi_y(V_i) \cap \pi_y(V_j)$ is a single point if $j = i + 1$, and is empty otherwise.

Proof. Recall that V_0 has height

$$|V_0| = h = \frac{\lambda^{n+1}}{\lambda^{n+1} + 1}.$$

Since $F_n(1, 0) = (0, 1)$ and $|V_1| = h/\lambda$, it follows that

$$V_1 = \{0\} \times \left[1 - \frac{h}{\lambda}, 1\right].$$

Now, using (23),

$$F_n(0, 1) = \left(p_2, \frac{1}{\lambda} - \frac{1}{\lambda^{n+1} + 1}\right) = \left(p_2, 1 - \frac{h}{\lambda}\right).$$

That is, the top point of V_2 is on the same horizontal level as the bottom point of V_1 . Since F_n preserves horizontal and vertical lines in $x \leq 1 - 1/\lambda$, the top point of V_{i+1} is on the same horizontal level as the bottom point of V_i for $1 \leq i \leq n - 1$. Since

$$\sum_{i=1}^n |V_i| = h \sum_{i=1}^n \frac{1}{\lambda^i} = \frac{\lambda^{n+1} - \lambda}{\lambda^{n+2} - \lambda^{n+1} + \lambda - 1} = 1 \quad (\text{by (23)}),$$

statement b) of the lemma follows.

In particular,

$$V_n = \{p_n\} \times \left[0, \frac{h}{\lambda^n}\right],$$

and since $F_n(p_n, 0) = (p_{n+1}, 1)$ it follows that

$$V_{n+1} = \{p_{n+1}\} \times \left[1 - \frac{h}{\lambda^{n+1}}, 1\right].$$

Now $1 - h/\lambda^{n+1} = \lambda^{n+1}/(\lambda^{n+1} + 1)$, the vertical coordinate of the top of V_0 , so statement a) follows also. \square

Horizontal sides H_0, \dots, H_{n+1} can therefore be added to join the endpoints of the vertical sides and bound the polygon P_n . Specifically, let

- H_0 join the top of V_0 to the bottom of V_{n+1} ;
- H_i join the bottom of V_i to the top of V_{i+1} for $1 \leq i \leq n-1$;
- H_n join the bottom of V_n to the bottom of V_0 , and
- H_{n+1} join the top of V_{n+1} to the top of V_1

(see Figure 14). In particular, H_{n-1} is the only horizontal side which intersects the line of discontinuity $x = p_{n+1} = 1 - 1/\lambda$ in its interior.

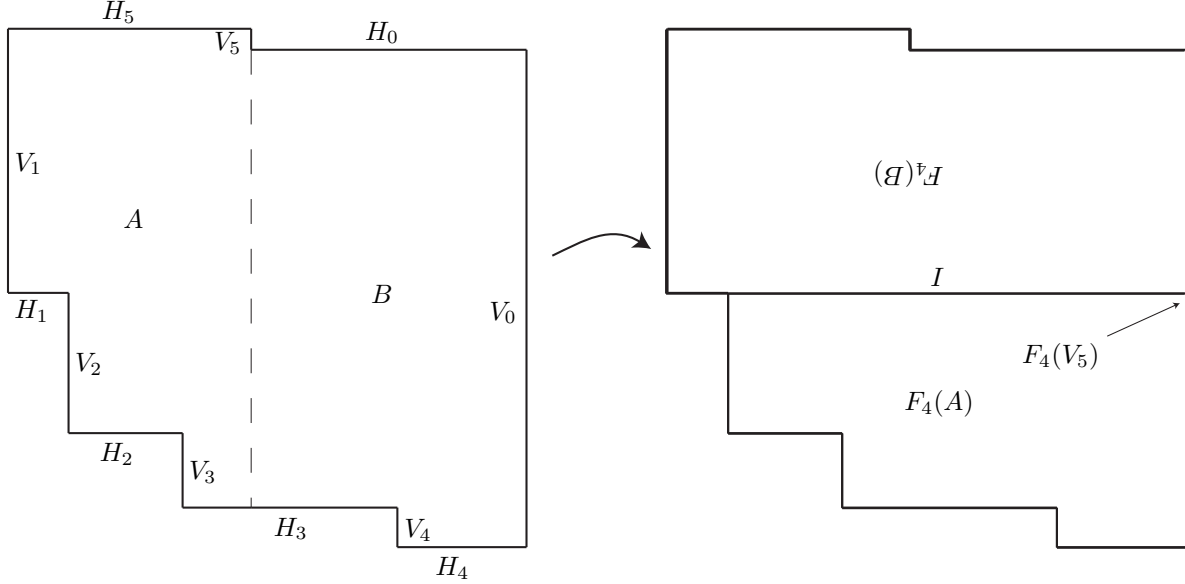


FIGURE 14. The polygon P_4 and its image under F_4

Next, the segment pairings on ∂P_n which make F_n continuous and injective will be described. The pairings on the vertical sides are straightforward and are treated first.

Lemma 95. *The image under F_n of the midpoint of V_{n+1} is the midpoint of V_0 .*

Proof. The midpoint of V_{n+1} has vertical coordinate $1 - h/2\lambda^{n+1}$, and so its image has vertical coordinate

$$\frac{1 - h/2\lambda^{n+1}}{\lambda} - \frac{1}{\lambda^{n+1} + 1} = \frac{1}{\lambda} - \frac{1}{2\lambda(\lambda^{n+1} + 1)} - \frac{1}{\lambda^{n+1} + 1}.$$

Using (23), this is equal to $\frac{\lambda^{n+1}}{2(\lambda^{n+1} + 1)}$, the vertical coordinate of the midpoint of V_0 . \square

Define segment pairings on the vertical sides of P_n by folding each side V_i about its midpoint. Since V_i is mapped affinely onto V_{i+1} for $0 \leq i \leq n$, and V_{n+1} is mapped affinely into V_0 with the midpoint being sent to the midpoint, the identifications arising from these pairings are respected by the action of F_n . Moreover, the identifications make F_n continuous across the line $x = 1 - 1/\lambda$.

Write \tilde{P}_n for the quotient space under these identifications on the vertical sides (a disk with boundary $\bigcup H_i$), so that F_n induces a continuous surjection, $\tilde{F}_n: \tilde{P}_n \rightarrow \tilde{P}_n$.

The purpose of the segment pairings on the horizontal sides of P_n is to make \tilde{F}_n injective: they are more complicated to describe. Observe first how F_n acts on these sides:

- $F_n(H_0) = I \cup H_1$, where I is the horizontal segment joining the right hand end of H_1 to V_0 (see Figure 14). Write $H_0 = H'_0 \cup H''_0$, where $F_n(H'_0) = I$ and $F_n(H''_0) = H_1$;
- $F_n(H_i) = H_{i+1}$ for $1 \leq i \leq n-2$;
- $F_n(H_{n-1}) = H_n \cup H_0$ (recall that H_{n-1} crosses the line of discontinuity $x = 1 - 1/\lambda$);
- $F_n(H_n) = H_{n+1}$; and
- $F_n(H_{n+1}) = I$.

In order to make F_n injective, it is therefore necessary to identify H_{n+1} with H'_0 , and then to propagate this identification under the dynamics.

Writing $I \rightarrow J$ to mean $F_n(I) \supset J$, observe that

$$H''_0 \rightarrow H_1 \rightarrow H_2 \rightarrow \cdots \rightarrow H_{n-1} \rightarrow H''_0.$$

There is therefore a period n point q_0 of F_n in H''_0 , the points $q_i = F_n^i(q_0)$ ($1 \leq i < n$) of whose orbit satisfy $q_i \in H_i$, and are therefore ordered cyclically according to their indices around the boundary of \tilde{P}_n .

Lemma 96. *The horizontal coordinate of q_0 is $1 - \frac{2-\lambda}{\lambda(\lambda+1)}$. Moreover, the distance from the left hand endpoint of H_0 to q_0 is equal to the length of H_{n+1} plus the distance from the left hand endpoint of H_1 to q_1 .*

Proof. Let L be the segment of H_0 from its left hand end to q_0 , and M be the segment of H_1 from q_1 to its right hand end (see Figure 15). Then $F_n(L) = I \cup M$ and $F_n^{n-1}(M) = L$. Thus

$$\lambda|L| = |I| + |M| = \lambda - 1 + |M|$$

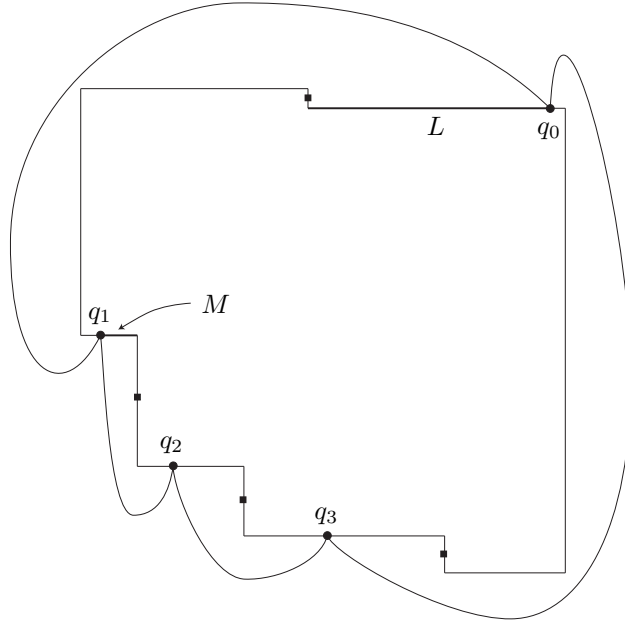
(using that the right hand endpoint of H_1 has horizontal coordinate $2 - \lambda$), and

$$\lambda^{n-1}|M| = |L|.$$

These give $|L| = \lambda^{n-1}(\lambda - 1)/(\lambda^n - 1)$, and hence the horizontal coordinate ξ of q_0 , given by $\xi = 1 - \frac{1}{\lambda} + |L|$, is

$$\xi = 1 - \frac{1}{\lambda} + \frac{\lambda^{n-1}(\lambda - 1)}{\lambda^n - 1}$$

which simplifies to the required expression using (23).

FIGURE 15. Identifications on the horizontal sides of P_4 .

Now $|H_{n+1}| = 1 - 1/\lambda$, and the distance from the left hand endpoint of H_1 to q_1 is

$$|H_1| - |M| = 2 - \lambda - \frac{|L|}{\lambda^{n-1}} = 2 - \lambda - \frac{\lambda - 1}{\lambda^n - 1},$$

and the sum of these simplifies, using (23), to $|L|$ as required. \square

It follows that identifying L with the union of H_{n+1} and the segment of H_1 between its left hand endpoint and q_1 identifies H_{n+1} with H'_0 as required. Propagating this identification under the dynamics requires identifying the two halves of the segment of the boundary of \tilde{P}_n between q_i and q_{i+1} for each $0 \leq i < n - 1$, and the two halves of the segment between q_{n-1} and q_0 (Figure 15). Each of these identifications can be realised as a segment pairing on P_n , with the exception of the identification of L with H_{n+1} and a segment of H_1 , and the identification of the two halves of the segment between q_{n-1} and q_0 , each of which can be expressed as two segment pairings. There is therefore a total of $n + 2$ horizontal segment pairings, in addition to the $n + 2$ vertical segment pairings.

The paper surface $S_n = P_n/d_{P_n}^{P_n}$ is a sphere, on which the horizontal and vertical foliations of P_n descend to a pair of transverse measured foliations, with $n + 2$ 1-pronged singularities (at the midpoints of the vertical sides) and an n -pronged singularity (at the points of the period n orbit Q on the horizontal boundary, which are all identified). Since F_n stretches in the horizontal direction by a factor λ , and contracts in the vertical direction by a factor $1/\lambda$, it induces a pseudo-Anosov homeomorphism $f_n: S_n \rightarrow S_n$ of the paper surface.

The scar G_n of (P_n, \mathcal{P}_n) is a tree with $2n + 4$ edges corresponding to the $2n + 4$ segment pairings in \mathcal{P}_n (Figure 16): however, n of its vertices (the projections of the endpoints of V_i for $2 \leq i \leq n + 1$) have valence 2. It will be convenient to describe it in the following way. Let $q_0 \in G_n$ denote the projection of the periodic orbit $\{q_0, \dots, q_{n-1}\}$ in P_n : q_0 will be called the *central vertex* of G_n . There are n *horizontal edges* h_0, \dots, h_{n-1} of G_n which emanate from q_0 and have lengths $|h_i| = \alpha\lambda^i$, where $\alpha = (\lambda - 1)/(\lambda^n - 1)$ so that $\sum_{i=0}^{n-1} \alpha\lambda^i = 1$. Similarly there are $n + 2$ *vertical edges* v_0, \dots, v_{n+1} of G_n , which have lengths $|v_i| = \beta\lambda^i$, where $\beta = (\lambda - 1)/(\lambda^{n+2} - 1)$. For $0 \leq i < n$, the edge v_i emanates from the end of h_{n-1-i} . On the other hand v_n (corresponding to V_1) emanates from h_{n-1} at distance α/λ from q_0 (this distance comes from Lemma 96); and v_{n+1} (corresponding to V_0) emanates from h_{n-2} at distance α/λ^2 from q_0 . In particular, h_{n-2} and h_{n-1} are strictly speaking unions of two edges of G_n .

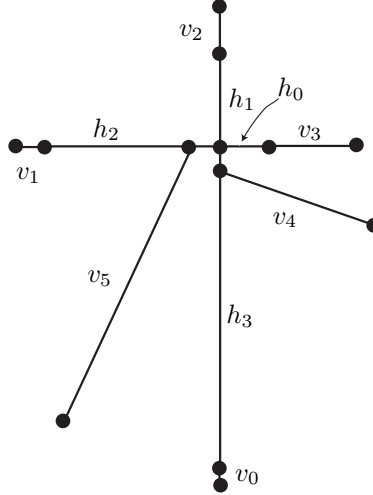


FIGURE 16. The scar of the P_4 paper surface

The final lemma of this section shows that the polygons P_n converge to the square Σ (see Figure 17), the equivalence relations $\sim_{\mathcal{P}_n}$ converge to the equivalence relation $\sim_{\mathcal{P}}$ of the tight horseshoe, and the functions $F_n: P_n \rightarrow P_n$ converge in an appropriate sense to $F: \Sigma \rightarrow \Sigma$. From this point on, superscripts n will be added to indicate dependence on n where they were previously omitted: thus, for example, the horizontal and vertical sides of P_n will be denoted H_i^n and V_i^n for $0 \leq i \leq n + 1$.

Lemma 97.

- a) Let $\varepsilon > 0$. Then the square $\Sigma_\varepsilon = [\varepsilon, 1 - \varepsilon] \times [\varepsilon, 1 - \varepsilon]$ is contained in P_n for all sufficiently large n . In particular, the polygons P_n converge in the Hausdorff metric to Σ .
- b) The equivalence relations $\sim_{\mathcal{P}_n}$ converge in the Hausdorff metric to the equivalence relation $\sim_{\mathcal{P}}$.

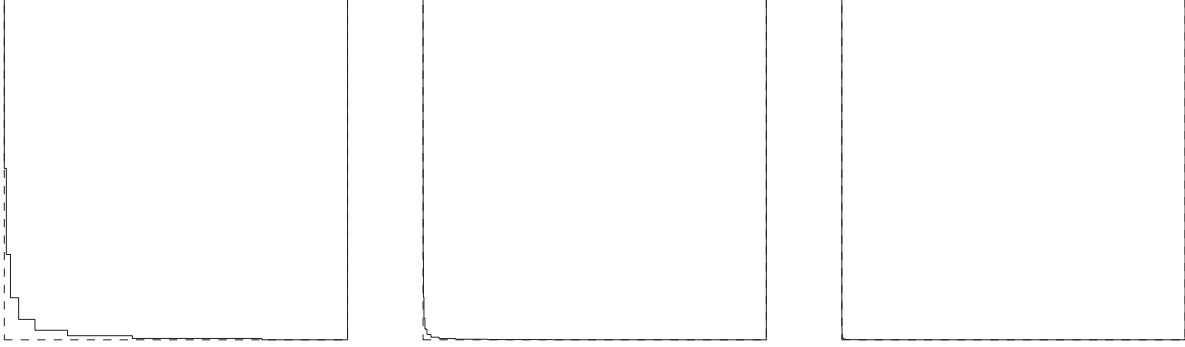


FIGURE 17. The polygons P_n (here $n = 8, 12, 16$) converge to Σ (shown with dashed lines)

- c) Let $(x_n, y_n) \rightarrow (x, y)$ be a convergent sequence in Σ with $(x_n, y_n) \in P_n$ for all n . If $x \neq 1/2$ then $F_n(x_n, y_n) \rightarrow F(x, y)$. If $x = 1/2$ then there is a subsequence (x_{n_i}, y_{n_i}) such that $F_{n_i}(x_{n_i}, y_{n_i}) \rightarrow z \in \Sigma$ where $z \sim_{\mathcal{P}} F(x, y)$.

Proof.

- a) It is enough to show that all of the sides of P_n are contained in $C_\varepsilon = \Sigma \setminus \Sigma_\varepsilon$.

V_0^n, V_1^n, H_n^n , and H_{n+1}^n , being subsets of $\partial\Sigma$, are contained in C_ε for all n .

Since $\lambda_n > 3/2$ for all n (as $\lambda_3 \simeq 1.722$), the heights of the vertical sides satisfy $|V_i^n| < (2/3)^i$. Pick K large enough that $\sum_{i=K}^\infty (2/3)^i < \varepsilon$: then the vertical side V_i^n is contained in $[0, 1] \times ([0, \varepsilon] \cup (1 - \varepsilon, 1]) \subset C_\varepsilon$ whenever $i > K$: this in turn means that $H_i^n \subset C_\varepsilon$ whenever $i \geq K$, and $H_0^n \subset C_\varepsilon$ provided that $n > K$.

Now pick $N > K$ large enough that $2 - \lambda_N < \varepsilon/2^{K+1}$, and suppose that $n \geq N$. Then, by (24), whenever $2 \leq i \leq K$ the side V_i^n has horizontal coordinate

$$\begin{aligned} p_i^n &= \frac{(2 - \lambda_n)(\lambda_n^{i-1} - 1)}{\lambda_n - 1} \\ &< 2(2 - \lambda_n)(\lambda_n^{i-1} - 1) \quad (\text{as } \lambda_n > 3/2) \\ &< \varepsilon(\lambda_n^{i-1} - 1)/2^K < \varepsilon, \end{aligned}$$

and hence $V_i^n \subset [0, \varepsilon] \times [0, 1] \subset C_\varepsilon$: this in turn means that $H_{i-1}^n \subset C_\varepsilon$.

- b) Let $\varepsilon > 0$. It is required to show that there is some N such that, for all $n \geq N$:

- i) If $a, b \in \Sigma$ with $a \sim_{\mathcal{P}} b$, then there exist $a', b' \in P_n$ with $a' \sim_{\mathcal{P}_n} b'$ and $|a - a'| < \varepsilon$, $|b - b'| < \varepsilon$.
- ii) If $a, b \in P_n$ with $a \sim_{\mathcal{P}_n} b$, then there exist $a', b' \in \Sigma$ with $a' \sim_{\mathcal{P}} b'$ and $|a - a'| < \varepsilon$, $|b - b'| < \varepsilon$.

There are four cases to consider:

Case 1: $a = b$. This is dealt with by a).

Case 2: a and b are an interior pair of a vertical segment pairing.

Choose K so that $\sum_{i=1}^K |V_i^n| > 1 - \varepsilon/2$ for all $n > K$. For each i with $0 \leq i \leq K$, $|V_i^n| = \lambda_n^{n+1-i}/(\lambda_n^{n+1} + 1)$ converges to $1/2^i$ as $n \rightarrow \infty$. Thus, for sufficiently large n :

- V_0^n is within Hausdorff distance $\varepsilon/2$ of $\{1\} \times [0, 1]$;
- V_i^n is within Hausdorff distance $\varepsilon/2$ of $\{0\} \times [1/2^i, 1/2^{i-1}]$ for $1 \leq i \leq K$; and
- $|V_i^n| < \varepsilon/2$ for $K < i \leq n+1$.

That is, each folded vertical segment in P_n (respectively Σ) is either very close to a folded vertical segment in Σ (respectively P_n), or is very small. In the former case, (a, b) can be approximated by some (a', b') ; and in the latter case, it can be approximated by some (a', a') .

Case 3: a and b are an interior pair of a horizontal segment pairing.

The argument is similar to that of the second case, except that (see Figure 15) the relevant quantities are the horizontal coordinates of the points $\{q_0^n, q_1^n, \dots, q_{n-1}^n\}$ of the periodic orbit of F_n on ∂P_n . When n is large, the horizontal coordinate of q_0^n is very close to 1, and the horizontal coordinates of q_i^n for $1 \leq i < n$ are either very small, or very close to $1/2^{n-i}$.

Case 4: a and b are endpoints of segment pairings.

The argument is an extension of the third case. When n is large, the set of segment endpoints on $\partial \Sigma$ is Hausdorff close to the union of the periodic orbit of F_n on ∂P_n with the points $(1, 0)$, $(0, 1)$ of ∂P_n : these two points are very close to being identified with q_0^n and q_1^n respectively.

- c) If $x < 1/2$ (respectively $x > 1/2$) then $x_n < 1 - 1/\lambda_n$ (respectively $x_n > 1 - 1/\lambda_n$) for all sufficiently large n , and the result is immediate from the definitions of F_n and F .

If $x = 1/2$ then $F(x, y) = (1, y/2)$. Take a subsequence (x_{n_i}, y_{n_i}) such that either $x_{n_i} < 1 - 1/\lambda_{n_i}$ for all i , or $x_{n_i} > 1 - 1/\lambda_{n_i}$ for all i . Then $F_{n_i}(x_{n_i}, y_{n_i})$ converges to $(1, y/2)$ in the former case, and to $(1, 1 - y/2) \sim_{\mathcal{P}} (1, y/2)$ in the latter case.

□

7.3. Polygon constants. Recall (Definition 68) that the polygon constants of a polygon P are: the length $|\partial P|$ of its boundary; the collaring height $\bar{h} > 0$ (Section 5.3.1); and $\bar{r} > 0$, which in the plain case can be chosen arbitrarily. In this section it is shown that these constants can be chosen uniformly for the polygons P_n . Since $|\partial P_n| = 4$ for all n , and \bar{r} is arbitrary, the only issue is to find a uniform collaring height \bar{h} . It will be shown that $\bar{h} = \frac{1}{24}$ is a collaring height for all n , and the constant \bar{r} will then be chosen as

$$\bar{r} = \bar{h} = \frac{1}{24}.$$

Observe (Figure 18) that the trapezoid of height $\bar{h} = 1/24$ on every side of P_n except for V_0^n , V_1^n , H_n^n , and H_{n+1}^n is a parallelogram. However, the lengths of these four exceptional sides are

uniformly bounded below:

$$\begin{aligned}
|V_0^n| &= \frac{\lambda_n^{n+1}}{\lambda_n^{n+1} + 1} > \frac{1}{2}, \\
|V_1^n| &= \frac{|V_0^n|}{\lambda_n} > \frac{1}{4}, \\
|H_{n+1}^n| &= 1 - \frac{1}{\lambda_n} > \frac{1}{3}, \quad \text{and} \\
|H_n^n| &= \frac{|H_{n+1}^n|}{\lambda_n} > \frac{1}{6}.
\end{aligned}$$

Hence the ratio between the lengths of the bases and the tops of the trapezoids on these sides lies in $[1/2, 2]$ since $\bar{h} \leq \frac{1}{6} \times \frac{1}{4} = \frac{1}{24}$. (Since the base angles are $\pi/4$, the difference between the length of the base and the length of the top is $2\bar{h}$.) It therefore only remains to show that the trapezoids of height \bar{h} only intersect along their common vertical sides. This is an immediate consequence of the observation that the difference between the vertical coordinates of the top of V_0^n and the bottom of V_1^n is

$$\frac{\lambda_n^{n+1}}{\lambda_n^{n+1} + 1} - \left(1 - \frac{\lambda_n^n}{\lambda_n^{n+1} + 1}\right) = \frac{1 - 1/\lambda_n^n}{\lambda_n + 1/\lambda_n^n} > \frac{1 - (2/3)^3}{2 + 1} = \frac{19}{81} > 2\bar{h}.$$

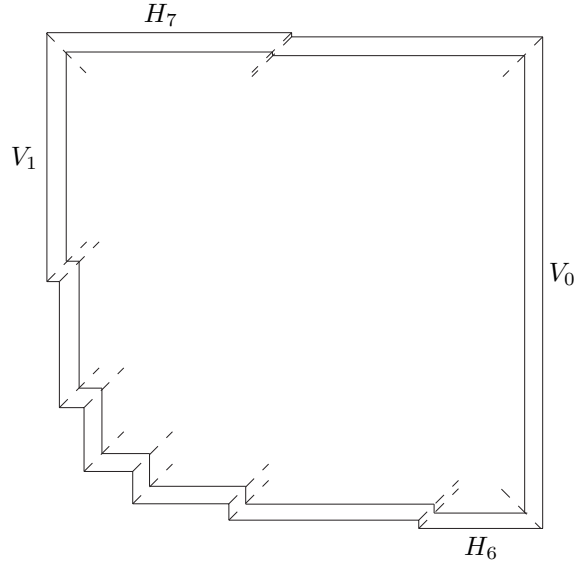


FIGURE 18. Collaring P_n (here $n = 6$)

7.4. Convergence of the uniformizing maps. Since the scars G_n have no singular points, the paper spheres S_n all have a natural complex structure, and there are unique uniformizing maps $u_n: S_n \rightarrow \widehat{\mathbb{C}}$ such that the compositions $\phi_n = u_n \circ \pi_n: P_n \rightarrow \widehat{\mathbb{C}}$ satisfy $\phi_n(1, 0) = 0$,

$\phi_n(1/2, 1/2) = \infty$, and $(1/\phi_n)'(1/2, 1/2) = 1$ (the point $(1/2, 1/2)$ lies in the complement of the height \bar{h} collaring of P_n for sufficiently large n by Lemma 97a): in fact it can be shown that this is true for all $n \geq 3$). The pseudo-Anosov homeomorphisms $f_n: S_n \rightarrow S_n$ induce homeomorphisms $\hat{f}_n: \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ by $\hat{f}_n = u_n \circ f_n \circ u_n^{-1}$.

Similarly, recalling that (Σ, \mathcal{P}) is the paper-folding scheme used in the definition of the tight horseshoe $f: S \rightarrow S$, there is a unique uniformizing map $u: S \rightarrow \hat{\mathbb{C}}$ with the property that the composition $\phi = u \circ \pi: \Sigma \rightarrow \hat{\mathbb{C}}$ satisfies $\phi(1, 0) = 0$, $\phi(1/2, 1/2) = \infty$, and $(1/\phi)'(1/2, 1/2) = 1$. The generalized pseudo-Anosov $f: S \rightarrow S$ induces a homeomorphism $\hat{f}: \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$.

In this section it is shown that the functions ϕ_n converge to ϕ , and hence that the homeomorphisms \hat{f}_n converge to \hat{f} . The key result, Lemma 100 below, is that the ϕ_n have a uniform modulus of continuity.

Definition 98 ($\mathcal{I}(t)$). Let $\mathcal{I}: (0, \bar{r}) \rightarrow (0, \infty)$ be the function defined by

$$\mathcal{I}(t) := \frac{\ln 2}{12} \int_t^{\bar{r}} \frac{ds}{s \ln \left(\frac{8}{s-t} \right)}.$$

Remark 99. \mathcal{I} is a decreasing function, having derivative

$$\mathcal{I}'(t) = \frac{-\ln 2}{12} \int_t^{\bar{r}} \frac{ds}{s(s-t) (\ln(8/(s-t)))^2} < 0.$$

Moreover $\mathcal{I}(t) \rightarrow \infty$ as $t \rightarrow 0$.

Lemma 100. Define $\bar{\rho}: [0, \delta) \rightarrow [0, \infty)$ by

$$\bar{\rho}(t) := \begin{cases} 0, & \text{if } t = 0; \\ \max \left(\frac{16R}{\exp(2\pi M \mathcal{I}(2At))}, \kappa t \right), & \text{if } t > 0. \end{cases}$$

Then for all $n \geq 3$, $\bar{\rho}$ is a modulus of continuity for $\phi_n: P_n \rightarrow \hat{\mathbb{C}}$, with respect to the Euclidean metric on P_n and the spherical metric on $\hat{\mathbb{C}}$.

Here $\delta = 1/4608$ and $A = 96$ are given by (20), $M = 1/5$ is given by Definition 77, $R = R(\bar{h})$ is given by Lemma 83, and $\kappa = \kappa(|\partial P_n|, \delta)$ is given by (22). Observe that $\bar{\rho}$ is a modulus of continuity in the sense of Definition 79: it is continuous, positive, and strictly increasing in $(0, \delta)$ as the maximum of two functions with these properties; and $\bar{\rho}(t) \rightarrow 0$ as $t \rightarrow 0$ by Remark 99.

Two preliminary lemmas are required.

Lemma 101. Let $n \geq 3$ and $r \in (0, \bar{r})$. Then, in G_n ,

$$\begin{aligned} m(q_0; r) &\leq 8r \log_2(8/r) & \text{and} \\ n(q_0; r) &\leq 4 \log_2(4/r), \end{aligned}$$

where q_0 is the central vertex of G_n .

Proof. The edges $v_0, v_1, v_n, v_{n+1}, h_{n-1}$ and h_{n-2} of G_n contribute at most $8r$ to $m(q_0; r)$ and at most 4 to $n(q_0; r)$. The remaining edges form an $(n-2)$ -od, with edges of lengths

$$\ell_i = |h_i| + |v_{n-1-i}| = \alpha\lambda^i + \beta\lambda^{n-1-i} \quad (0 \leq i < n-2).$$

Here $\alpha = \frac{1}{1+\lambda+\dots+\lambda^{n-1}} < \frac{1}{\lambda^{n-1}}$ and $\beta = \frac{1}{1+\lambda+\dots+\lambda^{n+1}} < \frac{1}{\lambda^{n+1}}$, so that

$$\ell_i < \frac{1}{\lambda^{n-1-i}} + \frac{1}{\lambda^{i+2}} < \frac{1}{(\sqrt{2})^{n-1-i}} + \frac{1}{(\sqrt{2})^{i+2}} := L_i.$$

Let $k = 2 + \lceil 2 \log_2(1/r) \rceil \leq 3 + 2 \log_2(1/r)$ (where $\lceil x \rceil$ denotes the smallest integer which is not less than x), so that $\frac{1}{(\sqrt{2})^k} \leq \frac{r}{2}$. Then provided $n-1-i \geq k$ and $i+2 \geq k$ (i.e. $k-2 \leq i \leq n-1-k$), L_i satisfies

$$L_i = \frac{1}{(\sqrt{2})^{n-1-i}} + \frac{1}{(\sqrt{2})^{i+2}} \leq \frac{2}{(\sqrt{2})^k} \leq r.$$

There are at most $2k-4$ values of i which don't satisfy these inequalities, so that

$$n(q_0; r) \leq 4 + (2k-4) \leq 6 + 4 \log_2(1/r) < 4(2 + \log_2(1/r)) = 4 \log_2(4/r)$$

as required. Similarly

$$m(q_0; r) \leq 8r + (4k-8)r + 2 \sum_{i=k-2}^{n-1-k} L_i < 8r + (4k-8)r + 8r < 8r(3 + \log_2(1/r)) = 8r \log_2(8/r),$$

since

$$\sum_{i=k-2}^{n-1-k} L_i < \frac{2}{(\sqrt{2})^k} \sum_{i=0}^{\infty} \frac{1}{(\sqrt{2})^i} < \frac{8}{(\sqrt{2})^k} \leq 4r.$$

□

Lemma 102. For all $n \geq 3$, $q \in G_n$, and $t \in (0, \bar{r})$

$$\int_t^{\bar{r}} \frac{ds}{m(q; s) + s \cdot n(q; s)} \geq \mathcal{I}(t).$$

Proof. Write $D := d_{G_n}(q_0, q)$, where q_0 is the central vertex of G_n . Then

$$m(q; r) \leq \begin{cases} 6r, & \text{if } r \leq D; \\ 6D + m(q_0; r-D) \leq 6D + 8(r-D) \log_2\left(\frac{8}{r-D}\right) & \text{if } r > D, \end{cases}$$

and

$$n(q; r) \leq \begin{cases} 3, & \text{if } r \leq D; \\ 2 + n(q_0; r-D) \leq 2 + 4 \log_2\left(\frac{4}{r-D}\right) < 4 \log_2\left(\frac{8}{r-D}\right) & \text{if } r > D. \end{cases}$$

There are three cases to consider: $0 \leq D \leq t$; $t \leq D \leq \bar{r}$; and $D \geq \bar{r}$.

Case 1: $0 \leq D \leq t$

Then

$$\int_t^{\bar{r}} \frac{ds}{m(q; s) + s \cdot n(q; s)} \geq \int_t^{\bar{r}} \frac{ds}{6D + (12s - 8D) \log_2 \left(\frac{8}{s-D} \right)}.$$

Since $\log_2(8/(s-D)) \geq \log_2(8/\bar{r}) > 1$, this integral is in turn bounded below by

$$F(D, t) := \int_t^{\bar{r}} \frac{ds}{12s \log_2 \left(\frac{8}{s-D} \right)} = \frac{\ln 2}{12} \int_t^{\bar{r}} \frac{ds}{s \ln \left(\frac{8}{s-D} \right)}.$$

Now

$$\frac{\partial F}{\partial D} = \frac{-\ln 2}{12} \int_t^{\bar{r}} \frac{ds}{s(s-D) \left(\ln \left(\frac{8}{s-D} \right) \right)^2} < 0$$

for $D \in [0, t]$, so

$$\int_t^{\bar{r}} \frac{ds}{m(q; s) + s \cdot n(q; s)} \geq F(t, t) = \mathcal{I}(t)$$

for all $q \in G_n$ with $d_{G_n}(q, q_0) \leq t$ as required.

Case 2: $t \leq D \leq \bar{r}$

Then

$$\int_t^{\bar{r}} \frac{ds}{m(q; s) + s \cdot n(q; s)} \geq \int_t^D \frac{ds}{9s} + F(D, D) = \frac{1}{9} \ln \left(\frac{D}{t} \right) + \frac{\ln 2}{12} \int_D^{\bar{r}} \frac{ds}{s \ln \left(\frac{8}{s-D} \right)}.$$

The derivative of this lower bound with respect to D is

$$\begin{aligned} \frac{1}{9D} - \frac{\ln 2}{12} \int_D^{\bar{r}} \frac{ds}{s(s-D) \left(\ln \left(\frac{8}{s-D} \right) \right)^2} &\geq \frac{1}{D} \left(\frac{1}{9} - \frac{\ln 2}{12} \int_D^{\bar{r}} \frac{ds}{(s-D) \left(\ln \left(\frac{8}{s-D} \right) \right)^2} \right) \\ &= \frac{1}{D} \left(\frac{1}{9} - \frac{\ln 2}{12 \ln \left(\frac{8}{\bar{r}-D} \right)} \right) \\ &> \frac{1}{D} \left(\frac{1}{9} - \frac{\ln 2}{12 \ln(192)} \right) > 0 \end{aligned}$$

for $D \in [t, \bar{r}]$, using $\bar{r} = 1/24$. Hence

$$\int_t^{\bar{r}} \frac{ds}{m(q; s) + s \cdot n(q; s)} \geq \frac{1}{9} \ln(1) + F(t, t) = \mathcal{I}(t)$$

for all $q \in G_n$ with $t \leq d_{G_n}(q, q_0) \leq \bar{r}$ as required.

Case 3: $D \geq \bar{r}$

Then

$$\int_t^{\bar{r}} \frac{ds}{m(q; s) + s \cdot n(q; s)} \geq \frac{1}{9} \int_t^{\bar{r}} \frac{ds}{s} > \frac{\ln 2}{12} \int_t^{\bar{r}} \frac{ds}{s \ln\left(\frac{8}{s-t}\right)} = \mathcal{I}(t)$$

as required, using $\ln(8/(s-t)) > \ln(8/\bar{r}) > 1$. \square

Proof of Lemma 100. Following the proof of Theorem 84, it is only required to show that the function $\rho_q(t)$ of (21) satisfies $\rho_q(t) \leq \bar{\rho}(t)/2$ for all n , all $q \in Q_n(\delta)$, and all $t \in (0, \delta)$. (As in that proof, the factor 2 arises from the translation between the Euclidean and spherical metrics.)

So let $q \in Q_n(\delta)$ and $t \in (0, \delta)$. If $h_q \leq t$ then

$$\begin{aligned} \rho_q(t) &= \frac{8R}{\exp\left(2\pi \int_{A(t+h_q)}^{\bar{r}} \iota(\psi(q); s) ds\right)} \\ &\leq \frac{8R}{\exp\left(2\pi \int_{2At}^{\bar{r}} \iota(\psi(q); s) ds\right)} \\ &\leq \frac{8R}{\exp(2\pi M\mathcal{I}(2At))} \end{aligned}$$

by Lemma 102, using $\iota(q; s) = \frac{M}{m(q; s) + s \cdot n(q; s)}$.

On the other hand, if $h_q > t$ then an analogous argument gives

$$\rho_q(t) \leq \frac{8Rt}{h_q \exp(2\pi M\mathcal{I}(2Ah_q))}.$$

However the function $x \mapsto x \exp(2\pi M\mathcal{I}(2Ax))$ is increasing on $(t, \delta]$, so

$$\rho_q(t) \leq \frac{8Rt}{t \exp(2\pi M\mathcal{I}(2At))}$$

as required. \square

Lemma 103. *For each $N \geq 3$ the sequence of functions $(\phi_n|_{X_N} : X_N \rightarrow \widehat{\mathbb{C}})_{n \geq N}$ converges uniformly to $\phi|_{X_N} : X_N \rightarrow \widehat{\mathbb{C}}$, where $X_N = \bigcap_{n \geq N} P_n$. In particular, if $x_n \rightarrow x$ is a convergent sequence in Σ with $x_n \in P_n$ for all n , then $\phi_n(x_n) \rightarrow \phi(x)$.*

Remark 104. The sequence of polygons (P_n) is not increasing, since the lengths $|V_1^n| = \lambda_n^n / (\lambda_n^{n+1} + 1)$ of the sides V_1^n , which are contained in $\partial\Sigma$, decrease with n . However, the sequence (X_n) is increasing, and converges Hausdorff to Σ by Lemma 97a).

Proof. Since the sequence (ϕ_n) is equicontinuous and the sequence of domains (P_n) converges Hausdorff to Σ , a variant of the Arzelà-Ascoli theorem (which is proved identically to the standard version) shows that there is a subsequence (ϕ_{n_i}) and a continuous function $\phi_\infty : \Sigma \rightarrow \widehat{\mathbb{C}}$ such that,

for each $N \geq 3$, the sequence $(\phi_{n_i}|_{X_N})_{n_i \geq N}$ converges uniformly to $\phi_\infty|_{X_N}$. It will be shown that $\phi_\infty = \phi$ for any such subsequence, which will establish the result.

Now $\phi_\infty(1/2, 1/2) = \infty$ and $\phi_\infty(1, 0) = 0$, since $\phi_n(1/2, 1/2) = \infty$ and $\phi_n(1, 0) = 0$ for all n . Moreover, because the functions ϕ_n are univalent on $\text{Int}(P_n)$ and any open subset of $\text{Int}(\Sigma)$ whose boundary is disjoint from $\partial\Sigma$ is contained in X_n for sufficiently large n , the function ϕ_∞ is univalent in $\text{Int}(\Sigma)$, and satisfies $(1/\phi_\infty)'(1/2, 1/2) = 1$. Because ϕ_∞ restricted to $\text{Int}(\Sigma)$ is open and injective, there do not exist points $w \in \partial\Sigma$ and $z \in \text{Int}(\Sigma)$ with $\phi_\infty(w) = \phi_\infty(z)$.

By Lemma 97b), if $w, z \in \partial\Sigma$ with $w \sim_{\mathcal{P}} z$, then $\phi_\infty(w) = \phi_\infty(z)$ ((w, z) is arbitrarily closely approximated by points $(w_n, z_n) \in \sim_{\mathcal{P}_n}$, which therefore satisfy $\phi_n(w_n) = \phi_n(z_n)$). There is therefore a continuous function $u_\infty: S \rightarrow \widehat{\mathbb{C}}$ with $\phi_\infty = u_\infty \circ \pi$, which restricts to a conformal homeomorphism

$$u_\infty|_{S \setminus G}: S \setminus G \rightarrow \widehat{\mathbb{C}} \setminus u_\infty(G).$$

Let $z \in \partial\Sigma$ be any point in the interior of a paired segment α of \mathcal{P} , let $\varepsilon > 0$ be the distance from z to the nearest endpoint of α , and let z' be the point of the paired segment α' which is paired with z . Let $\Sigma^z \subset \mathbb{C}$ be obtained from Σ by excising $B_\Sigma(z'; \varepsilon/2)$ and gluing this half-disk onto Σ near z according to the pairing $\langle \alpha, \alpha' \rangle$. Then ϕ_∞ induces a natural function $\phi_\infty^z: \Sigma^z \rightarrow \widehat{\mathbb{C}}$.

Choose sequences $z_n \rightarrow z$ and $z'_n \rightarrow z'$ where $z_n, z'_n \in P_n$ and $z_n \sim_{\mathcal{P}_n} z'_n$. Then for sufficiently large n an analogous construction can be carried out to obtain functions $\phi_n^z: P_n^z \rightarrow P_n^z$ which are univalent on $\text{Int}(P_n^z)$. It follows that ϕ_∞^z is univalent on $\text{Int}(\Sigma^z)$, and in particular that there is no point $w \in \partial\Sigma$ with $\phi_\infty(w) = \phi_\infty(z)$ other than z' .

Now if $w, z \in G$ with $w \neq z$ and $u_\infty(w) = u_\infty(z)$, then every point except possibly one of the interval $[w, z]_G$ is identified with another such point by u_∞ , as otherwise $u_\infty(G)$ would contain a simple closed curve, contradicting the fact that its complement is connected; in particular, u_∞ identifies some point in the interior of a paired segment with another point which it is not $\sim_{\mathcal{P}}$ -equivalent to. Since this has been shown to be impossible, u_∞ is injective on G .

It follows that $u_\infty: S \rightarrow \widehat{\mathbb{C}}$ is a homeomorphism: since it is conformal on $S \setminus G$, and G has finite 1-dimensional Hausdorff measure by Lemma 33c), u_∞ is conformal by Theorem 27. Because u_∞ is normalized in the same way as u the two are equal, and hence

$$\phi_\infty = u_\infty \circ \pi = u \circ \pi = \phi$$

as required. □

Theorem 105. *The pseudo-Anosov homeomorphisms $\widehat{f}_n: \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ converge pointwise to the generalized pseudo-Anosov $\widehat{f}: \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$.*

Proof. Let $z \in \widehat{\mathbb{C}}$ and choose, for each n , a point $x_n \in P_n$ with $\phi_n(x_n) = z$: thus $\widehat{f}_n(z) = \phi_n(F_n(x_n))$. It will be shown that if $\phi_{n_i}(F_{n_i}(x_{n_i})) \rightarrow w$ is any convergent subsequence then $w = \widehat{f}(z)$, which will establish the result.

Given such a subsequence, assume without loss of generality that $x_{n_i} \rightarrow x \in \Sigma$, and (using Lemma 97c)) that $F_{n_i}(x_{n_i})$ converges to some $y \in \Sigma$ with $\phi(y) = \phi(F(x))$. Since $\phi_{n_i}(x_{n_i}) = z$ for all i , it follows from Lemma 103 that $\phi(x) = z$. By the same lemma,

$$\phi_{n_i}(F_{n_i}(x_{n_i})) \rightarrow \phi(y) = \phi(F(x)) = \hat{f}(\phi(x)) = \hat{f}(z)$$

as required. □

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DEPARTAMENTO DE MATEMÁTICA APLICADA, IME - USP, RUA DO MATÃO 1010, CIDADE UNIVERSITÁRIA,
05508-090 SÃO PAULO, SP, BRAZIL

E-mail address: `andre@ime.usp.br`

DEPARTMENT OF MATHEMATICAL SCIENCES, UNIVERSITY OF LIVERPOOL, LIVERPOOL L69 7ZL, UK

E-mail address: `T.Hall@liv.ac.uk`